ELEMENTARY
EXPERIMENTAL DYNAMICS
FOR SCHOOLS

CAMBRIDGE UNIVERSITY PRESS
London: FETTER LANE, E.C.
C. F. CLAY, Manager

Edinburgh: 100, PRINCES STREET
Berlin: A. ASHER AND CO.
Leipzig: F. A. BROCKHAUS
New York: G. P. PUTNAM'S SONS
Bombay and Calcutta: MACMILLAN AND CO., Ltd.

ELEMENTARY EXPERIMENTAL DYNAMICS FOR SCHOOLS

BY

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Cambridge:
at the University Press
1913

Cambridge:

PRINTED BY JOHN CLAY, M.A.

AT THE UNIVERSITY PRESS

PREFACE

THIS book forms the second part of an introductory course of Mechanics for schoolboys. To add to the number of text-books on Dynamics calls for apology, or at least for an indication that the guiding principles differ to some extent from those usually followed; hence a brief statement of the point of view adopted may be advisable.

The ordinary deductive treatment of kinetics, starting from Newton's Laws as axioms, though satisfactory for mature minds, has proved ill-suited to young boys. Inductive methods appeal to them with much greater force, and it seems advisable that their attention should be concentrated at first on simple quantitative experiments and the development of the fundamental principles of mechanics from their results.

These principles involve ideas so difficult and so unfamiliar that it is highly inexpedient to develop them by means of mathematical processes which are too recently acquired to be instinctive. Even if practice in the application of such processes is the real object in teaching mechanics, the final result will be better if the mathematics used in the early stages is limited to arithmetic and very simple numerical trigonometry or geometry. Methods which are more explicitly mathematical may be introduced when the pupil has advanced some distance on these lines, and he will then be in a position to appreciate their generality and their advantages as labour-saving devices. At a later stage, of course, he can make no progress without the use of formal mathematics, but it is desirable that he should retain even then the habit of realising the mechanical meaning of each step in the manipulation of the symbols. To quote Thomson and Tait, "Nothing can be more fatal to progress than a too confident reliance on mathematical symbols; for the student is only too apt to take the easier course, and consider the *formula* and not the *fact* as the physical reality."

This numerical work in no way saves a boy from the necessity of thinking; in fact, the refusal to supply him with formulae and standard methods forces him to trust to his own powers, and reveals to him, as well as to his teacher, any failure in comprehension. With this object, numerical examples have been interspersed throughout the text; questions involving descriptions have been kept separate in order that they may be combined as desired with the numerical questions. The introduction of more general methods is deferred almost to the end of the book, so that the master may be free to use them earlier with such pupils as are ready for them, without detriment to those who are slower.

The course is designed so that all the more important experiments can be performed by the master with very few pieces of apparatus in an ordinary mathematical class-room; but it is obviously preferable that the pupil should perform some at least of them. A few details of suitable apparatus are given in an appendix.

In order to instil concrete ideas of mass, work, energy, momentum, etc., many of the illustrations have been drawn from simple engineering practice. The modern boy is keenly interested in such machines as motor-bicycles and aeroplanes, and is sufficiently familiar with them on their qualitative side not to require elaborate descriptions of their mode of action; his sense of power is greatly increased when he finds that his study of mechanics enables him to get even approximate values for their performances. The policy of refusing to touch anything until it can be dealt with completely may commend itself to the cautious teacher, but it is very cramping to the growing mind; for example, it appears preferable to give a broad idea of the principles underlying the action of a screw propeller and a rough numerical approximation to its behaviour under given conditions, rather than to wait until some future mathematician shall have discovered a method of solving the problems it presents. When the boy passes from the class-room to the larger world he will find that progress does not always wait until the theory is complete, and it is a good thing to accustom him early to use such knowledge as he possesses in obtaining the best results within his reach, and thus to realise the desirability of further knowledge to render these results more trustworthy. This principle has been widely adopted in the teaching of elementary pure mathematics, and there is every reason for extending it in the study of mechanics.

There is a large number of text-books on elementary Applied Mechanics, written for students of Engineering, which deal in this manner with real rather than academic problems, but many teachers feel that they pay so little attention to logical treatment that they are better adapted to technical instruction than to education; a serious effort has been made in this little book to maintain an adequate standard in this respect.

In the elementary treatment of a subject of such antiquity, plagiarism is inevitable and often unconscious, and its due acknowledgment in individual cases becomes impossible; but the author is under an especial debt of gratitude to his colleagues at the Royal Naval College, Dartmouth, for permission to draw largely on their experience, and on the store of examples which they have made for their own use; more particularly to Mr Portway, who has checked the answers to the examples. He also wishes to express his acknowledgment to the publishers of *Engineering* for permission to reproduce Figs. 73 and 78.

C. E. A.

DARTMOUTH,
March, 1913.

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CHAPTER I

VELOCITY

1. Suppose we have a rifle weighing 7 lbs., which fires a bullet weighing 1 oz. at a velocity of 2000 feet per second, and we wish to calculate the force it will exert on a shoulder which yields to the kick through a distance of 5 inches. Or suppose that we can supply steam at a pressure of 100 lbs. per square inch to the cylinders of a locomotive (of which we know the size of every part), and we wish to calculate how long it will take to get up a speed of 40 miles an hour when it starts from rest and drags a train weighing 100 tons up an incline of 1 in 150. These are examples of problems in Dynamics. You will notice that they bring in various quantities, velocities, lengths, times, forces, &c.; also that the same kind of quantity may be expressed in different units, as velocity in "feet per second" or "miles an hour." As we go on we shall find that all the quantities we meet with can be referred back to three "fundamental" ones, **Length, Time** and **Mass.**

In Statics we were not concerned with Time or Mass but only with lengths and forces; we treated Force as a fundamental quantity, but we shall see that Force can be referred back to length, time and mass in the same way as other quantities. So it will be well to clear the ground by briefly considering these three quantities.

2. Length. Length usually comes into dynamics as the distance through which a body, or a point in a body, moves when forces act on it. To describe the movement fully, we require to know its starting or finishing point, as well as the distance between these points, and its direction. As everything is in motion, including the earth and probably the solar system, we really have no fixed point or fixed direction in space to reckon from, but in ordinary life we treat the earth as a fixed body and it is convenient to do so in dynamics, except when we deal with the motion of the heavenly bodies. For simplicity we shall at first consider only the case of bodies which do not rotate, i.e. which move so that the line joining any two points in the body remains parallel to itself throughout the motion, as for example in a train running along a straight piece of line.

Time. Although it is difficult to give a definition of time, every one knows what is meant by it; we usually measure it by a watch indicating seconds or fifths of a second. Such a watch is generally regulated so that it shows $24 \times 60 \times 60$ of these seconds between noon on one day and noon on the next, and the seconds are then called "mean solar seconds"; if we regulated it by the stars instead of the sun the seconds would be very slightly shorter; seconds are "mean solar seconds" if nothing is said to the contrary.

But we often have to measure intervals of time which are shorter than a fifth of a second; this we can do by means of a vibrating spring, as will be explained in Art. 4.

Mass. The same force applied to different bodies will produce different results—you can for example discover whether an egg is empty or not by shaking it to and fro. Every body contains a certain "quantity of matter"; this phrase may have various meanings, but in dynamics it has the meaning that most people would give to it, that the quantity of matter in a body is large if it is heavy, and small if it is light; in fact, that the quantities of matter in two bodies are compared by their weights, whatever materials they are made of. As dynamics deals with the effect of forces on bodies, and weight is a force, this is clearly a reasonable way of comparing "quantities of matter" in dynamics; we shall discuss later some other ways in which the quantity of matter in a body is measured for other purposes, and shall show that so far as dynamics is concerned, weight is the proper basis for comparing quantities of matter. To make it clear that we are thinking only of dynamics, we use the special name, **Mass,** for "the quantity of matter in a body" whenever we have to deal with the effect of forces on the body.

But the fact that the weight of a body and the quantity of matter in it are measured in the same way leads to great confusion between them in the mind of the ordinary person; this confusion is not very harmful in everyday life, but in dynamics it is essential to bear constantly in mind the fact that they are entirely different quantities. The Mass of a body is the quantity of stuff in a body, its Weight is the force with which the earth attracts it. Whenever you meet with the word Mass you should call to mind this distinction, although you probably will not, for some time, see why it is so important.

3. Average speed. If a man spends an hour in walking three miles, we say his average speed has been three miles an hour; he may have kept on walking steadily for the whole time at the same speed of three miles an hour, or he may have stopped somewhere on the way for a quarter of an hour, and walked the three miles in the remaining three quarters of an hour, during which time his actual speed must have averaged four miles an hour. In getting an average speed we take into account only the total distance covered and the total time occupied, and we divide the former by the latter to get the numerical value of the speed. There is no generally accepted unit in which speeds are expressed, but they are stated in miles an hour, feet per second, &c., as may be most convenient. For example the speed of a train or motor car would commonly be expressed in miles per hour or kilometres per hour, that of the piston of a steam-engine in feet per second, that of a conductor in the armature of a dynamo in centimetres per second. But if the speed of a body is known in any given units of length and time, it is easy to calculate what it will be when expressed in any other units. It should also be noted that we talk of a train going at an average speed of 30 miles per hour even if it has not travelled for a whole hour; for example, if it covers 10 miles in 20 minutes.

Ex. 1. A train covers 88 feet in a second; what is its average speed in miles per hour?

If it kept up the same average speed for an hour, it would go 60×60 times as far, that is $60 \times 60 \times 88$ feet, or $\frac{60 \times 60 \times 88}{1760 \times 3}$ miles, or 60 miles; so its average speed is **60 miles an hour.**

This fact, that 60 miles an hour is the same speed as 88 ft. per sec., occurs so often in calculations about speeds of trains, &c., that it is convenient to remember it.

Ex. 2. In $\frac{1}{8}$th of a second a body is observed to move through 1·4 cm.; express its average speed in cm. per sec., and in feet per sec.

As before, in 1 sec. if it maintained the same average speed it would go through $8 \times 1{\cdot}4$ cm., or 11·2 cm.; and since

$$1 \text{ in.} = 2{\cdot}54 \text{ cm.}, \quad 11{\cdot}2 \text{ cm.} = \frac{11{\cdot}2}{2{\cdot}54} \text{ ins., or } \frac{11{\cdot}2}{2{\cdot}54 \times 12} \text{ ft., or } {\cdot}367 \text{ ft.}$$

So its average speed is **11·2 cm. per sec.** or **·367 ft. per sec.**

Ex. 3. Which has the faster average speed, a pigeon at 1600 yards per minute or a train at 55 miles per hour?

Ex. 4. A motor car is timed over 220 yards as taking 22 secs.; what is its speed in miles per hour? Supposing the distance may be 4 yds. wrong, and the time $\frac{2}{5}$ sec. wrong, between what limits does the real speed lie?

Ex. 5. A man walks $\frac{1}{2}$ mile at 4 miles per hour, waits five minutes for a train; travels in it at 40 miles per hour for 12 miles; drives at 10 miles per hour for 15 minutes. Find his average speed, and hence find whether he would do the whole distance more quickly in a motor at 20 miles per hour.

Ex. 6. In four successive runs over a measured mile, a ship covered the distance in 2 mins. 14 secs., 2 mins. 30 secs., 2 mins. 8 secs., and 2 mins. 27 secs. What was her average speed in miles per hour for each run? What was the average speed for the four miles? Is this latter the same result as you get by finding the average of the times in which the mile was covered, and deducing an "average speed" from that; or is it the average of the speeds for the different miles?

Ex. 7. A gun is fired at you from a fort 4 miles off. Taking the speed of light to be 186,000 miles a sec., the speed of sound to be 1100 ft. a sec., and the average speed of the shot to be 1500 ft. a sec., at what intervals will the flash, the report of the gun, and the shot reach you?

4. Measurement of small intervals of time. Example 2 suggests that in a laboratory we can measure the distance travelled by a body in a smaller interval of time than is recorded by a stop watch; we will now explain how it can be done.

If a flat steel spring is clamped at one end, and the other end is pulled aside and let go, the spring will vibrate for some little time. The vibrations gradually die down, the *distance* through which the end of the spring moves getting less and less, but the *time* occupied by each vibration does not decrease. This "periodic

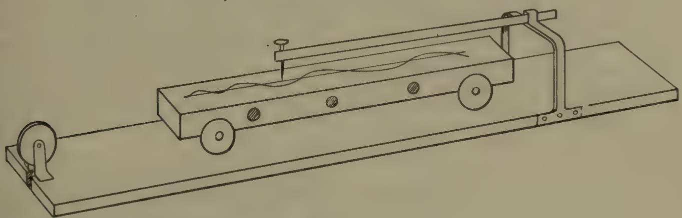


Fig. 1.

time" of the spring, as it is called, depends on the length, breadth and thickness of the spring and the quality of the steel, but does not depend on the size, or "amplitude," of the oscillations, provided that these are not large. This statement can be verified by direct counting against a stop watch in the case of a spring which makes 3 or 4 vibrations a second. But it requires some practice to count rapidly enough, even at this speed, and the following method of verifying it will serve as an introduction to an arrangement of apparatus which we shall often use.

We shall want a flat strip of steel about two feet long, clamped at one end so as to lie horizontally, with its edge upwards, above a horizontal surface of wood or iron on which a carriage can run smoothly. The spring carries at its free end a paint-brush which lightly touches a strip of paper fixed to the top of the carriage. This carriage or trolley is long and narrow—say 2 ft. by 3 ins., and runs on three or four small wheels. If the paint brush is filled with ink and the carriage is pushed along while the spring is at rest, a straight line is drawn on the paper strip; but if the spring is now set in motion and the carriage is slowly pushed along, we get a wavy line which in future we shall call a "tracing," marked on the paper, see Fig. 2. If the spring is started with fairly large vibrations, and the carriage is pushed along during a known number of seconds (fifteen or twenty), as shown by a watch, the number of vibrations which the spring made in this time can be counted at leisure. The experiment is repeated with the spring making vibrations of smaller size, and the number of vibrations per second in each

Fig. 2.

case will probably be found to be nearly the same. This experiment also gives us the time of one vibration of the spring, as accurately as we are likely to need to know it in our experiments. Suppose for convenience that the spring is adjusted so that it makes 5 vibrations a second; this adjustment can be made by shortening or lengthening the part of the spring beyond the clamp. By a "vibration" we shall always mean a complete to-and-fro movement of the spring, so the time of one vibration, or the periodic time of the spring, means the time that elapses between the instant when the brush crosses the centre line and the instant when it crosses it again *moving in the same direction*; so the interval between consecutive crossings of the centre line is half the time of one vibration. It is always better to reckon by whole vibrations, since it is not then of great importance that the centre line should be truly central; if you look at a tracing where the "centre line" is not central you will understand the reason for this.

So by measuring the length along the centre line between alternate points of crossing of the wavy line (which length we will call "one wave-length") we know how far the carriage travelled during that fifth of a second, and can deduce its average speed for that interval. For example, in Fig. 2 the distance from A to B was found to be 9·8 cm., so the average speed there was $\frac{9 \cdot 8}{0 \cdot 2}$ or 49 cm. per sec.

This method can be applied to investigate the motion of a piston (by attaching the paper strip to the piston rod) or of a body falling under its own weight (by letting the carriage slip down vertically between guides), &c.

5. Uniform speed. It is a common expression, that a train is running "at a uniform speed," and you probably know what you mean by it, but would not find it easy to express its meaning precisely. You might explain it by saying that its speed was not getting faster or slower; but then you would have to explain the meaning of the train's speed "at an instant," and that also would be difficult to do. In practice, motor cars are provided with speedometers which mark the speed at any instant, and if the needle remains steady, you know that the speed is constant; but these do not help us to a definition, though they help us to connect the sensation of a certain speed with its measure in miles per hour.

Suppose you have a trolley running with very little friction on a horizontal plane, and give it a sharp push; it will seem to move with fairly uniform speed. Arrange a spring to make a tracing on it as it runs underneath, and find the average speed during an early and a late vibration of the spring, by measuring the distance travelled in one vibration. You will find that the average speed is less near the end than near the beginning of the motion, so the speed is clearly not quite uniform. Next raise one end of the plane, and get a tracing on the trolley after it has been started downhill with a push; you will probably find that the average speed during successive vibrations of the spring gradually increases. In the first case the trolley was retarded by friction, in the second the slope more than compensated the friction.

So it ought to be possible to find some angle of slope of the plane such that if the trolley is started down it, the average speed does not change; you can arrive at it by adjusting the slope until the trolley just goes on moving down when started gently, and you will find it easier to judge whether the speed is uniform if the speed is slow than if it is fast. When you are satisfied with the result as judged by eye, take a tracing when the carriage runs down fairly fast, and measure the average speeds during two or three separate vibrations, at the beginning, middle, &c., of the run; if necessary, further adjust the slope till the average speed so found does not vary. Whatever our definition, it is fairly clear that this must be uniform speed, so we will choose a working definition to fit this method of testing it, as follows: "*A body moves with uniform speed if its average speed during any interval is the same.*" This interval of time is to be of any length we choose to name, and to begin at any instant during the motion. This is not a perfect definition, and can be improved later.

A possible, if rather fanciful, example will show the need of trying several intervals of time during which you measure the average speed, before asserting that the speed is uniform. Suppose a motor travels along a regularly undulating road, the hills being all the same height and distance apart, and you take as your interval in each case the time from the top of one hill to the top of the next. Then you will always get the same value for the average speed, but the speed downhill is probably greater than that uphill. This will be detected by changing the interval, and timing the motor from the top to the bottom of a hill; if by no such changes can you find any different average speed, you are justified in thinking the speed is uniform.

6. Variable speed. Uniform speed is very exceptional in practice; bodies in motion generally vary their pace every instant, in consequence of changes in the conditions under which they are moving. In order to be able to realise at a glance how a body has moved, it is convenient to represent in a diagram the successive positions of the body at various instants of its motion. You are probably familiar with a "Distance-Time diagram" of this kind; the following extract from Bradshaw and the distance-time diagram which represents it graphically will serve as an illustration.

Lengths measured horizontally (called "abscissae") represent times, and above each minute so marked off lengths are measured vertically upwards (called "ordinates") to represent on a suitable scale the distance of the train from the starting point at the end of that minute. From this diagram or from the time-table we see that in the first five minutes it travelled $1\frac{1}{4}$ miles, so its average speed was $12 \times 1\frac{1}{4}$ or 15 miles an hour. From Acton to Ealing ($1\frac{3}{4}$ miles) took 5 minutes, so its average speed then was $12 \times 1\frac{3}{4}$ or 21 miles an hour. It will be seen that the line joining A to E is at a slightly steeper slope than that joining $M.H$ to $C.C$; and that on such a diagram the slope of the line joining two points is steeper the greater is the average speed between them. For the gradient of the line is measured by the rise in a unit horizontal distance; and when the latter represents a given time and the former the distance gone in the time, the average speed (which is the distance divided by the time) is represented by this gradient. Note, however, that we must take care to use, not the actual lengths on the diagram, but the distances or times which these lengths represent. Thus EM represents $1\frac{3}{4}$ miles and AM represents 5 mins., and the average velocity is expressed by the fraction $\frac{EM}{AM}$, when EM and AM have these meanings. It is important to note that the usual way of stating the gradient of a road or railway line is to give the distance *measured up the slope* in which there is a vertical rise of one unit length; for example,

Miles		mrn
	Mansion House..............dep.	5 20
½	Blackfriars	5 21
1	Temple....................................	5 23
1¼	**Charing Cross** (Embankment)	5 25
1¾	Westminster	5 26
2¼	St James' Park........................	5 28
2¾	**Victoria**	5 30
3¼	Sloane Square	5 32
4	South Kensington	5 34
4½	Gloucester Road	5 35
5	**Earls Court**	5 38
5¾	West Kensington	5 40
6	Barons Court	5 41
6½	**Hammersmith**	5 43
7	Ravenscourt Park	5 45
7½	Stamford Brook	5 46
8	Turnham Green	5 48
9	Gunnersbury	......
10	Kew Gardens	......
11¼	**Richmond**....................arr.	......
8½	Chiswick Park	5 50
9¼	**Acton Town** 409	5 52
10	Ealing Common	5 54
11	**Ealing** (Broadway) ....arr.	5 57

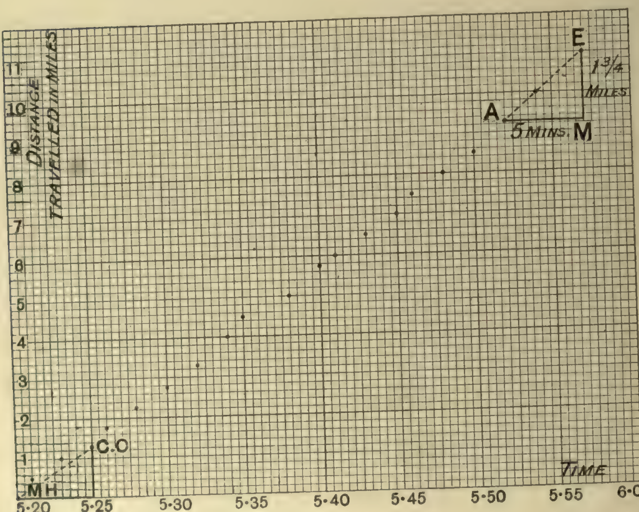


Fig. 3.

Fig. 4 shows a road with a gradient of 1 in 5. This method is obviously the most convenient for a surveyor, who can measure distances along the surface of the road with a chain.

But in dealing with diagrams or graphs, we always mean by "gradient" the *vertical rise of the curve in unit horizontal distance*; thus, the gradient of AB in Fig. 5 is $\frac{20}{8}$ or 2·5. The numbers 20 and 8 are not usually equal to the actual lengths of the lines BM and AM, but are the values of certain quantities represented by these lines on the scales used in making the diagram. This meaning of gradient is the most convenient in the case of graphs, for our measurements are then made along the edges of the squares of the squared paper.

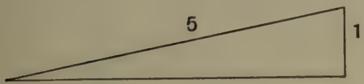


Fig. 4.

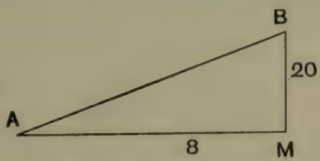


Fig. 5.

It should also be noted that Fig. 3 only gives us the position of the train at certain moments; we are not told where the train is at any other times than those marked on the diagram, and we are not justified in drawing a smooth curve through or near to these points, or even in joining the points by straight lines, in order to represent the position of the train at intermediate times. As a matter of fact, experience of trains tells us that they stop in each station for some time, start out slowly, gather speed until they are running steadily and then slow down for the next station. We could indicate this roughly on the diagram, but we should need to know a great deal more than we are told by Bradshaw before we could do so accurately; we can distinguish between what we know and what we estimate to be the case by putting in the latter as a dotted line.

Ex. 8. Draw a distance-time diagram for the following time-table:—

Miles			
0	Taunton	*a.* 4.6	
		d. 4.13	
$30\frac{1}{2}$	Exeter	*a.* 4.52	
		d. 4.57	
51	Newton Abbot...	*a.* 5.35	
		d. 5.42	
$82\frac{1}{2}$	Plymouth.........	6.28	

A convenient scale is 1 inch to 20 miles, and 1 inch to 20 mins.

Find from the diagram the average speed between stations in each case, by putting in *straight* dotted lines as *AE* in Fig. 3 above and finding the distances run in 20 minutes.

7. Motion of a carriage running freely from rest down a slope. If a trolley standing on an inclined plane is released, it is obvious that it moves with an increasing velocity, and it looks as though the velocity increases smoothly and not by

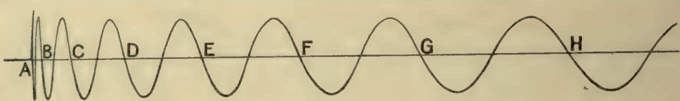


Fig. 6.

jerks. Take a tracing of the motion as before; it is convenient to use a trigger which releases the trolley when it is hit by the spring in its first passage across the centre line, for then the intervals of time begin from the instant at which the trolley starts from rest. Fig. 6 represents a tracing so produced.

The periodic time of the spring used was $\frac{1}{10}$ second. Measurement of the tracing gave the following values in cm. :—*AB*, ·7; *AC*, 2·7; *AD*, 6·1; *AE*, 11·4; *AF*, 17·8; *AG*, 25·5; *AH*, 34·7, &c. These results, plotted with a time-scale of 1 inch to $\frac{1}{10}$ sec., and a distance scale of 1 inch to 5 cm., gives the diagram of Fig. 7. In this case we are justified in drawing a smooth curve through the observed points to represent intermediate positions, as we saw that the increase of speed went on smoothly; so we can deduce with some certainty that after an interval of ·36 sec.

from the start the trolley had travelled over 9 cm. and so on.

The value of the average speed during any interval can be found as before; e.g. to find average speed during the third tenth of a second of the motion, draw the chord CD, and determine its gradient, which is $\frac{3 \cdot 4}{0 \cdot 1}$ or 34 cm. per sec. Similarly the average

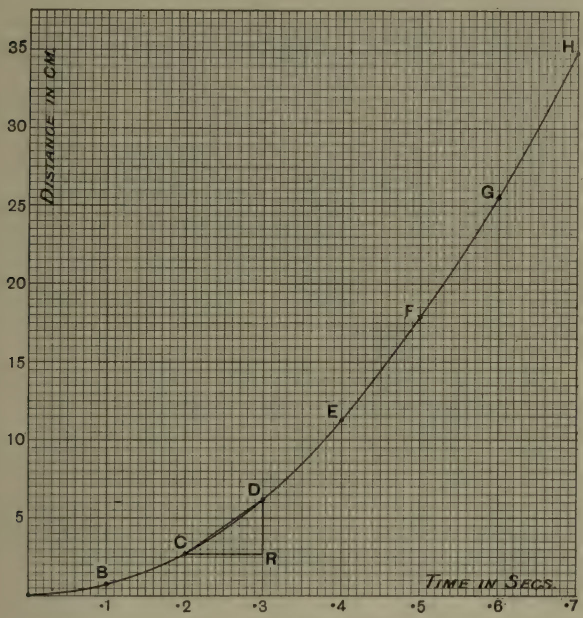


Fig. 7.

speed during any given interval of time, or over any part of the "run" can be determined by finding the gradient of the chord joining the corresponding points on the curve. For example, the average speed over the first 10 cm. is seen to be $\frac{10}{0 \cdot 38}$ cm. per sec.

8. Distance-time diagram for uniform speed. In Art. 5 we agreed that "when a body moves with uniform speed, its average speed during any interval is the same." If this is so, the gradient of every chord of the curve must be the same, and this can only be the case if the "curve" is a straight line having that gradient; so the distance-time diagram of a body moving at a uniform speed is a straight line.

9. Velocity at an instant. Take a trolley on a plane sloped at such an angle that the trolley when started runs down with uniform velocity, as described on page 8; it will be found that if this is the case for one speed it will be so for any other speed. Now attach a thread to the trolley, pass it over a pulley at the lower end of the plane so that the thread is parallel to the plane, and hang a small weight to the thread. When the trolley is released, the weight will pull it down the plane, and if a tracing is taken the distance-time diagram will be found to be similar to Fig. 7. Now adjust the length of the thread so that the small weight reaches the floor while the trolley is running under the spring, and take a tracing of the motion.

Pull the trolley back up the plane, until the weight is just being lifted from the floor, and mark on the tracing the position of the brush on the centre line. This gives the point in the trolley's run at which the weight ceased to act. In a case where this point was found to be 8 cm. from the starting point, the following were the measurements of the total distances travelled by the end of successive tenths of a second; ·7, 2·8, 6·4, 10·7, 15·4, 20·1 and 24·8 cm. Plot the distance-time diagram for this run, drawing as smooth a curve as you can through the observed points (Fig. 8). It will be seen that, as we should expect, the latter part of the curve is a straight line representing uniform speed. P is the point on the curve whose ordinate is 8 cm., and so it corresponds to the point at which the weight ceased to act. We see by the diagram that the *time* at which the weight ceased to act was ·34 sec. after the start. Since the speed did not change after that, it is reasonable to call this speed the "speed at this instant"; and this may be taken as the speed at this instant whether the weight hits the floor then or not. We can now understand the meaning of the phrase *the velocity of a body at a given instant*, as being *the uniform velocity with which the body would proceed if the applied forces ceased to act at that instant.*

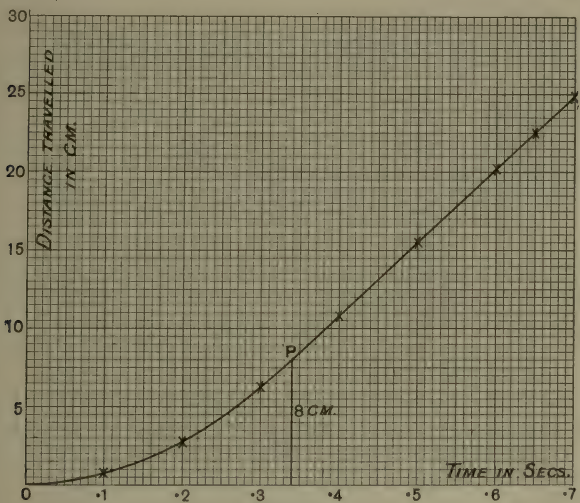


Fig. 8.

10. Speed and Velocity. It will be noticed that we have sometimes used the word Speed and sometimes Velocity, to represent the rate at which a body is moving. The word **speed** is commonly used when we are considering only the rate at which a body travels along its path, the word **velocity** when both speed and direction are considered. So the speed of a body moving in a circle, e.g. a stone in a sling, may be constant, but its velocity is continually changing because its direction of motion changes; and the velocity of a piston may alternate between zero and some large value, sometimes in one direction, sometimes in the opposite direction. We can distinguish between the two directions along a path by calling one the positive and the other the negative direction; then the sign attached to the number representing the velocity shows the direction in which the body is moving. When we are dealing with motion along one straight line and always in the same direction we can use either speed or velocity, and need not mention the sign.

Consider a stone being whirled round in a sling; its velocity changes from instant to instant, and its velocity at any instant is the velocity it would go on with if suddenly released at that instant, so that it "flies off at a tangent." The direction of the velocity at that instant is of course the direction of the tangent, and the magnitude of the velocity is the constant speed with which it was being whirled round in a circle before it was released.

Ex. 9. Calculate the value of the constant velocity of the trolley in the latter part of the experiment described in Art. 9, from the numbers given there. Calculate also the average velocity from the start up to the instant when the weight ceased to act. Calculate also the average velocities in each of the first three tenths of a second.

Ex. 10. A train starting out from a station and stopping at another $3\frac{3}{8}$ miles away is timed to pass successive quarter-mile posts, which start from the former station, at the following number of seconds from the start; 100, 141, 173, 200, 224, 245, 265, 286, 306, 326, 347, 367, 395. The run between stations took 450 secs. Plot the distance-time diagram of the run. The most convenient scales to adopt are 1 inch to half a mile and 1 inch to 100 secs. Find what was the speed of the train in miles per hour when it became uniform, the time at which you estimate this to have happened, and the average speed of the train for the whole run.

Miscellaneous Exercises.

Ex. 1. The distances from the starting point travelled by a body at the end of successive tenths of a second were observed to be: ·4, 1·22, 2·46, 2·86, 3, 3, 3·14, 3·5, 4, 4·5, 5, 5·5, 6 feet respectively. Plot the distance-time diagram, on scales of 1 in. to 1 ft. and 1 inch to ·2 sec. Determine the average speeds (1) for the whole time, (2) between ·6 sec. and 1 sec. from the start. Explain what the curve tells you about the speed of the body.

Ex. 2. An aeroplane and a motor car start together on a journey of 80 miles, the former going at 50 miles an hour and the latter at 15 metres a second, on the average. How many miles will the motor car be behind the aeroplane when the latter has finished its journey? (Take 10 metres = 11 yds.)

Ex. 3. A sailing boat crosses a lake to a point 5 miles due N. of the point of starting in 3 hrs. 20 mins., having to make 20 tacks each of length 1200 yards. Find (1) its average speed through the water, (2) its average velocity due North.

Ex. 4. A ship covers the measured mile in 2 mins. 24 secs. with the tide and in 4 mins. against it. Find its average speed over the ground in each trip and in the whole of the two miles. Also find its speed through the water, assuming that and the tide to be constant.

Ex. 5. A lift ascends a height of 100 ft. in 30 secs., what is its average speed? Its actual heights at the end of successive periods of 5 secs. are found to be: 6·7, 25, 50, 75, 93·3, 100 feet. Show the motion by a distance-time diagram.

Ex. 6. A man walks at 4 miles an hour for $\frac{1}{2}$ hour, and then walks 2 miles at three miles an hour. Find his average speed for the whole journey.

Ex. 7. The telegraph poles by a railway were 50 yds. apart, and a train was observed to pass a post every 4 secs. Calculate its speed in miles an hour.

Ex. 8. Assuming that the Earth is a sphere of 4000 miles radius, calculate the distance through which a person at the equator moves in one day; hence find his speed in miles an hour.

Ex. 9. Assuming that the orbit of the Earth round the Sun is a circle of radius 92,000,000 miles, find the Earth's average speed in its orbit, in miles an hour.

Ex. 10. A ship near the coast in a fog gives a short blast on its syren, and the echo from the cliffs is heard 6 secs. later. Assuming that sound travels at the rate of 1100 ft. per sec., calculate the ship's distance from the cliffs.

Ex. 11. London taxi-cab drivers can charge 2*d.* for each quarter-mile or each $2\frac{1}{2}$ minutes. How slowly must they drive in order to bring the latter rate of payment into operation while they are moving?

Ex. 12. A rough rule for converting speeds is to take two-thirds of the speed in ft. per sec. as the speed in miles per hour. What is the percentage error?

Ex. 13. An express train running at 65 miles an hour overtakes a local train 200 feet long which is running at 35 miles an hour; how long will it take a spectator in the former train to pass the latter?

Ex. 14. A man walks a distance of 14 miles in $3\frac{1}{2}$ hours; what is his average speed? If his average speed for the first 6 miles was 5 miles an hour, what was his average speed for the last 8 miles?

Ex. 15. A bristle is fixed to the end of a tuning fork which gives 256 vibrations a second, and pressed against a strip of smoked paper fastened on the rim of a rotating drum, the diameter of which is 6 ins. The wave lengths recorded are found to be of a constant length of ·16 in. Find the speed of the drum in revs. per minute.

Ex. 16. The driving wheels of a locomotive, which are 6 ft. in diameter, are making 126 revs. per minute. Calculate the speed of the train in miles per hour.

Ex. 17. During a passage of 7 hours a vessel steams at 6 knots for the first hour, at 8 knots for the next two hours, and at 10 knots for the last four hours. What is the average speed in knots for the whole 7 hours? Express this speed also in miles an hour. (1 knot = 6080 ft. per hour.)

Ex. 18. The volume swept out by the piston (single acting) of a single cylinder petrol motor is 40 cub. ins.; it makes 1200 revs. per min. and takes in a charge of gas at every other down stroke through a pipe 1 inch in diameter. Find the average speed of the gas in this pipe, expressed in ft. per sec. If this pipe supplies four such cylinders instead of one, find the speed of the gas in miles per hour.

CHAPTER II

THE EFFECT OF A CONSTANT FORCE ON A BODY

11. Inertia; Newton's First Law of Motion. Everyone knows from his ordinary experience that when a body is in motion, it "takes some stopping," and that a heavy weight "takes some starting." He also knows that it takes some little force to deflect a rapidly moving body into some other line of motion; for example, if he is skating fast and wishes to swing round and go on at right angles to his previous track, he has to exert considerable force on the ice to do it. We may include these three facts in one statement, that *force is* needed to change the velocity of a body. Other instances of the same general law may be cited. If you are standing up in a railway carriage when the train either starts or stops somewhat suddenly, you fall backwards or forwards unless you exert some force to prevent it by holding on to something; if the train is running fast round a curve, you feel yourself impelled towards the outer rail. A motor car or a bicycle making a sharp turn on a greasy road is liable to "skid," that is, to continue moving in the same direction as before. It needs a considerable push to start a "heavy" person sitting in a swing, though the person's *weight* does not come into play until he has swung through an appreciable angle; it is the quantity of matter to be set in motion that determines the force needed. A train runs on for some time after steam has been shut off the engine; it slows down and comes to rest because of friction. The smaller we make this retarding force of friction, the longer it takes to lose its speed; and if we could destroy the friction altogether, so far as we can tell it would never come to rest unless some outside force stopped it. This is the case with the motion of the moon and the planets; every change that occurs in their motion can be predicted from a knowledge of the force of attraction of the sun or earth, &c., on them, and we know of no instance in which they have started, or stopped, or changed their course without the help of some force outside themselves. These observations have been summed up in a general law by Sir Isaac Newton, which may be translated as follows: **"Every body continues in the same state of rest or of uniform motion in a straight line, except in so far as it may be compelled to change that state by applied forces."** This important law is called Newton's First Law of Motion.

The meaning of the phrase "applied forces" may not be clear at first sight. If a body is sliding over perfectly smooth horizontal ice, there will be "applied forces" on it, i.e. the attraction of the earth downwards and the pressure of the ice upwards; but these balance one another and are in equilibrium; they do not bring the body to rest. What is meant in this law is a force which is not balanced by another force acting on the same body, i.e. what is called a **resultant** force in statics.

A special name, **Inertia,** is given to the inability of matter to change its own state of rest or of uniform motion in a straight line. To overcome the inertia of a body an "applied" force is needed, and we know from experience that the amount of this applied force depends on the quantity of matter in the body, i.e. on its "mass," a larger force being needed to produce a given change of velocity in a large body than in a small one of the same material. The ordinary idea of the "massiveness" of a body is directly connected with the resistance it offers to a change in its velocity; if you wish to know whether a barrel lying on the ground contains much matter, you probably push it with your foot and judge by the resistance it opposes to you. So inertia is the name for a certain property of matter, and we use the word mass when we wish to express it numerically; other properties of matter are hardness, elasticity, &c., but they do not need to be expressed numerically so often as to make it worth while to invent a name, corresponding to mass, for their numerical measure.

12. Force. Our idea of force comes from our sensations when we use our muscles. If we hold a 14 lb. weight, we are conscious of exerting a definite force to prevent the weight from falling to the ground, and if we catch a cricket ball we are conscious of exerting a force to stop its motion. We say also that we can feel the weight, or the cricket ball, exerting a force on our hands. This latter force, which the earth exerts on the weight would set it in motion downwards, if we did not oppose it with another force; and the force which we exert on the weight would set it in motion upwards, if the attraction of the earth on the weight suddenly ceased. So both these forces tend to set the weight in motion, though they fail to do so. Again, the force exerted by the hand on the cricket ball stops its motion, and the force exerted by the cricket ball on the hand sets the hand in motion. We can sum up these conclusions in the statement, "*Force is that which changes, or tends to change, a body's state of rest, or of uniform motion in a straight line.*" This definition of force is merely another way of writing Newton's First Law of Motion.

As we cannot accurately compare forces by our sensations, we generally measure them by the "dead weight" they will support, or by the extension which they will produce in the spring of a spring balance. Either method will give the numerical value of the force to be measured, but it should be noted that the second depends on the first, for the spring balance is graduated by hanging weights on to it.

13. Measurement of Forces in Statics and Dynamics. In Statics we are dealing with bodies at rest, and we usually measure forces by weights acting on the body by strings, which are passed over pulleys if we want forces inclined to the vertical. But if the weight, instead of keeping the body at rest, overcomes the other forces opposing it and makes the body begin to move, then the force it exerts on the body along the string becomes less than its own weight, and so the force exerted can no longer be measured by the weight of the mass attached to the string. To show that this is so, we use the other method of measuring forces which is sometimes adopted in statics in place of "dead weights," namely the extension of a spring balance; the accuracy of this method of measuring a force is not affected by the motion of the spring balance itself, because that motion does not affect the elasticity of the steel of which the spring is made, on which depends the extension for a given pull. If then we fix a sensitive spring balance to a trolley, and attach to it a thread carrying a mass, as described on page 14,

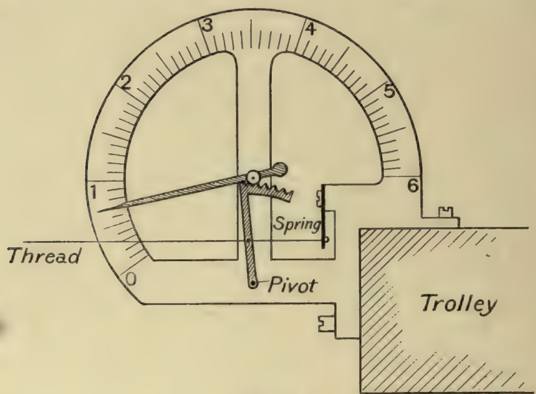


Fig. 9.

we can measure by the reading of the spring balance the force which the mass exerts on the trolley, first when the latter is held at rest, and then while the mass is allowed to make the trolley get up speed. We shall see that this force is less while the trolley is moving than when it is at rest, but that it remains constant during the motion whatever speed the trolley has attained. If we put different loads on the trolley, we shall find that the less the load the greater will be the difference between the force exerted by the mass when at rest and when allowed to move under the action of gravity. Here are the results of such an experiment. The acting mass was ·5 lb., and the spring balance recorded ·5 lb. wt. when the trolley was held at rest. When the trolley, loaded to 3 lbs., was released, the spring balance showed ·43 lb. wt. throughout the motion, a decrease of ·07 lb. wt.; when the trolley was loaded to a mass of 7 lbs. altogether, the spring balance showed ·47 lb. wt., a decrease of ·03 lb. wt. during the motion. So it would clearly lead us into serious errors if we assumed that the half-pound mass exerted a pull of half-a-pound weight on the trolley when they were moving; but we can take the pull as *constant*, whatever speed it produces. If we need to know its actual value we must use a spring balance, or calculate it by general laws to be discovered later; but in the early part of dynamics, all we require is that the force should remain constant throughout the motion which it produces, and we see that the pull exerted by a mass does this.

14. Effect of a constant force on a body initially at rest; Uniform Acceleration. We will now make experiments to determine how a body moves from rest when it is acted on by a force which is constant in magnitude and direction.

Slope the plane just enough to balance frictional resistances to the motion, so that the trolley when started down the plane runs on at constant speed. Attach to the trolley a thread passing over a pulley at the bottom of the plane and carrying a mass; take a tracing of the motion when the trolley is released. Determine the distances moved from rest up to each alternate transit of the brush across the centre line. The numbers found in a particular experiment were 1·52, 6·08, 13·68 and 24·32 cm. We do not need to know the time occupied in one vibration of the spring; we will call this time "one vibration."

Now arrange a platform at such a height that the mass just rests on it when the trolley has moved from rest through the first wave-length (1·52 cm. in our experiment), so that the force will cease to act after one vibration. Take a tracing of the motion under these conditions and determine from it the value of the uniform velocity which the trolley acquires, as on page 14; that is, the velocity at the end of one vibration from the start. This was found to be 3·04 cm. per vibration.

Move the platform to such a height that the mass rests on it after two vibrations (i.e. 6·08 cm. from the start in our experiment), and in the same way as before determine the velocity at the end of two vibrations; this was found to be 6·08 cm. per vibration. In the same way, determine the velocity acquired at the end of three vibrations; this was in our experiment found to be 9·12 cm. per vibration, and at the end of four vibrations it was found to be 12·16 cm. per vibration.

If these results are examined, it is seen that at the end of 1, 2, 3, etc. periods from the start the velocities are in the proportion of 1, 2, 3, etc. Since we have taken the period of vibration of the spring at random, it cannot be a mere coincidence that this is so; if there is any doubt, we can alter the period as often as we like and repeat the experiment, hence we are justified in saying that the velocity increases steadily with the time during which the constant force has acted. And as we have taken the load on the trolley and the constant force at random, we are justified in saying that the result is true of any body acted on by any force under these circumstances. So, *if a body moves from rest under a constant force, its velocity at any instant is proportional to the time elapsed from the start.* In other words, the body gets up speed steadily; its velocity increases each second by the same amount. When this is so, the amount by which the velocity of the body increases in a second is called its "*Acceleration*," and the body is said to move "with uniform acceleration."

In the above case the velocity increased by 3·04 cm. per vibration in each successive vibration, so the acceleration may be expressed as "3·04 cm.-per-vibration per vibration"; we cannot tell what is the increase of velocity *in a second* until we know the time of one vibration.

Fig. 10 is a combined distance-time diagram of the four tracings obtained in this experiment.

Ex. 1. In a similar experiment the velocity at the end of 4 vibrations from rest was found to be 7·85 cm. per vibration, and at the end of 6 vibrations

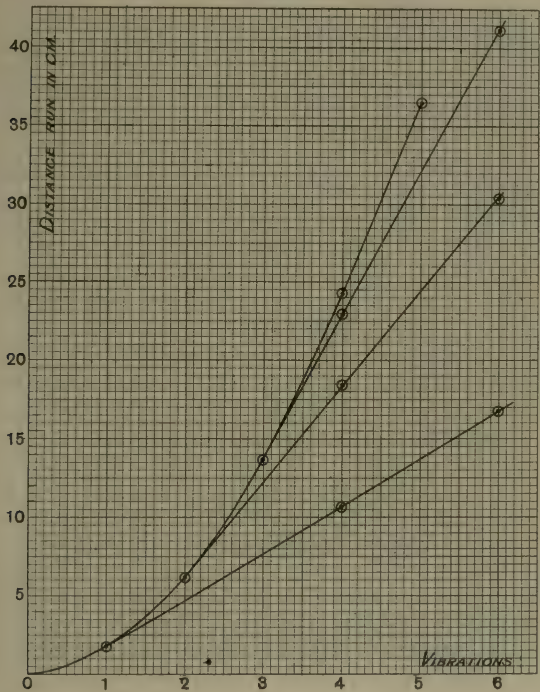


Fig. 10.

to be 11·77 cm. per vibration. Is the acceleration uniform, and if so, what is its value?

The rate at which velocity increased in the first 4 vibrations was $\frac{7{\cdot}85}{4}$, or 1·963, cm. per vibration per vibration, and in the first six it was $\frac{11{\cdot}77}{6}$, or 1·962, cm. per vibration per vibration.

Ex. 2. In a similar experiment, the velocity after 7 vibrations from rest was found to be 15·4 cm. per vibration. If the period of the spring is known to be $\frac{1}{5}$ sec., what is the value of the acceleration?

The observed velocity is 15·4 cm. per $\frac{1}{5}$ sec., or $5 \times 15{\cdot}4$ cm. per sec. This is acquired in $7 \times \frac{1}{5}$, or 1·4 secs.; at the same rate of increase, the velocity acquired in 1 sec. was $\frac{5 \times 15{\cdot}4}{1{\cdot}4}$, or 55·0, cm. per sec. Hence the acceleration is **55** cm. per sec. per sec.

Ex. 3. In a similar experiment, the velocity after 3 vibrations from rest was found to be 5·8 cm. per vibration. If the spring made 7 vibrations per second, what is the value of the acceleration?

Ex. 4. A train was known to be moving with uniform acceleration and at the end of 1 min. from starting its speed was 30 miles an hour. What was its speed at the end of 10 seconds from starting? How long would it take to reach a speed of 60 miles an hour?

15. The same tracings, or any other pair taken in the same way, will show another important fact about a body moving from rest under a constant force.

In our experiment we found that the trolley had travelled 1·52 cm. by the end of one vibration, and 24·32 cm. by the end of four vibrations. So the distance it travelled in the interval between these two instants was $24{\cdot}32 - 1{\cdot}52$ or 22·8 cm. Now the interval between the two instants was three vibrations, so the average velocity during the interval was $\frac{22{\cdot}8}{3}$ or **7·6** cm. per vibration.

We also found that its actual velocities at the beginning and end of this interval were 3·04 and 12·16 cm. per vibration respectively. The arithmetic mean of these two velocities is $\frac{3{\cdot}04 + 12{\cdot}16}{2}$ or **7·6** cm. per vibration; so we see that in this case, the average velocity during the interval is equal to half the sum of its velocities at the beginning and end of the interval.

As before, this cannot be mere coincidence, and we can take it as a general law that *if a body moves under a constant force, its average velocity during any interval is half the sum of its velocities* *at the beginning and end of the interval.* So in the case of a body moving with uniform acceleration, the average velocity is the "average" (as generally understood, i.e. the arithmetic mean) of its initial and final velocities; but this is not true unless the acceleration is constant.

Ex. 5. In a similar experiment, the distance travelled in 2 vibrations was found to be 4·24 cm., and the velocity acquired to be 4·24 cm. per vibration; at the end of 6 vibrations the distance from rest was found to be 38·16 cm. and the velocity acquired 12·72 cm. per vibration. Find the average speed and half the sum of the initial and final velocities for this interval; do these results verify the law given above?

Ex. 6. A train is running at 30 miles an hour and accelerates steadily during two minutes up to 50 miles an hour; how far does it run while doing so?

Its average speed is 40 miles an hour during the 2 minutes; hence the distance run is $1\frac{1}{3}$ miles.

16. If the body started from rest under the constant force, *the velocity at any instant is twice the average velocity from the start up to that instant*; for the initial velocity is zero, so half the sum of the initial and final velocities becomes half the final velocity.

Ex. 7. A body moves with uniform acceleration from rest and covers 1·06 cm. in the first second; what is its acceleration?

Its average speed in the first second is $\frac{1{\cdot}06}{1}$ or 1·06 cm. per sec.; so its speed at the end of the first second is $2 \times 1{\cdot}06$ or 2·12 cm. per sec.; so the acceleration (or increase of velocity per sec.) is **2·12** cm. per sec. per sec.

Ex. 8. A body moves from rest with uniform acceleration of 5·68 cm. per sec. per sec.; what speed does it attain in 5 secs., and how far does it run in that time?

Its velocity increases by 5·68 cm. per sec., so in 5 secs. from rest it is $5 \times 5{\cdot}68$ or **28·4** cm. per sec. Hence its average speed during this time is $\frac{28{\cdot}4}{2}$ or 14·2 cm. per sec.; and the distance run in 5 secs. at this average speed is $14{\cdot}2 \times 5$ or **71** cm.

Ex. 9. A body falls from rest with a uniform acceleration of 32 ft. per sec. per sec.; what will be its speed at the end of 5 secs., and how far will it have fallen?

Ex. 10. A body moves from rest with uniform acceleration, and covers 16·96 cm. in the first 4 secs.; find its speed at the end of that time, and deduce its acceleration.

Ex. 11. Calculate the speed of the body in Ex. 9, 9 secs. from the start; hence find its average speed during the interval from 5 to 9 secs. from the start; hence calculate the distance it falls during that interval.

Ex. 12. A trolley stands on a plane sloped at such an angle that it runs down with uniform speed. A mass is attached; the trolley is released by a trigger actuated by the spring; the mass reaches a fixed platform after descending 14·8 cm. The period of the spring is ·2 sec. The tracing gave the following results:

Time in secs.	0	·2	·4	·6	·8	1·0
Distance in cm.	0	·37	1·53	3·42	6·08	9·50
Time in secs.	1·2	1·4	1·6	1·8	2·0	2·2
Distance in cm.	13·68	18·21	22·95	27·70	32·46	37·20

Plot the distance-time diagram, on a scale of 1 in. to 10 cm., and 1 in. to ·4 sec. Determine from it the time after the start at which the force ceased to act (i.e. the time corresponding to 14·8 cm.), and the subsequent uniform speed. Calculate the average speed in cm. per sec. from rest up to this instant; is this half the subsequent uniform speed, i.e. the speed at the instant? Calculate the average speed from rest to the end of ·2, ·4, ·6, ·8, 1·0 and 1·2 secs.; hence (since acceleration is constant) find the speeds at the instants ·2, ·4, etc. secs. from the start. Plot these speeds and the subsequent uniform speeds on a diagram, taking times as abscissae on a scale of 1 inch to ·4 sec., and velocities as ordinates on a scale of 1 inch to 10 cm., as in Fig. 11. Account for the change of direction at the point P. Does this "velocity-time diagram" show that the speed increases uniformly while the force acts?

Ex. 13. Use the observations given in Art. 14 to calculate the average speed during each successive vibration from the start, expressed in cm. per vibration. Do these average speeds increase by the same amount from one vibration to the next?

Ex. 14. Use the observations given in Ex. 12 above, for distances travelled under the action of a constant force, to calculate the average speed during each of the first six successive fifths of a second. Do these average speeds increase by the same amount from one interval to the next, within the accuracy of observation (note that distances are measured only to the nearest tenth of a millimetre)?

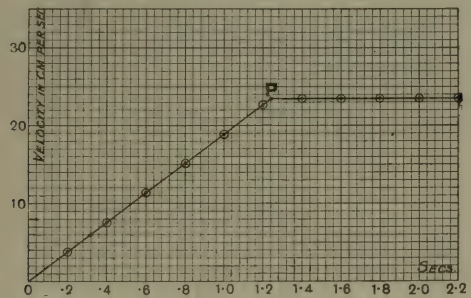


Fig. 11.

Ex. 15. The following are the measurements of a tracing taken on a piston-rod.

Time in fifths of a second	0	1	2	3	4	5	6	7	8	9
Distance in cm.	0	·15	·60	1·34	2·34	3·57	5·0	6·58	8·26	10·0

Plot the distance-time diagram, taking 1 inch to two-fifths of a sec., and 1 inch to 2 cm. Could you tell from inspection of the diagram whether the force was constant? Determine whether the average speed in successive fifths of a second increases by the same amount from one interval to the next, and compare with those of Ex. 13 and Ex. 14 above; do you think that the piston-rod was moving with uniform acceleration? The result of the comparison suggests a ready method of testing whether a body moves with uniform acceleration, by finding whether successive wave-lengths on a tracing increase uniformly.

17. Bodies falling freely from rest. It is not easy to determine the motion of a body falling freely under the action of its own weight alone; for, in the first place, the air resists the motion, to a greater or less extent according to the shape and density of the body; in the second place, the rate at which a falling body gets up speed is so great that in three seconds it is moving at about 60 miles an hour, so that we must use much smaller time intervals than we have been doing hitherto.

The first difficulty can to some extent be overcome by making observations before the body has got up much speed, for it is only at high speeds that the frictional resistance of the air is considerable. The second can be overcome by using, instead of a flat spring, one prong of a large tuning fork which makes, say,

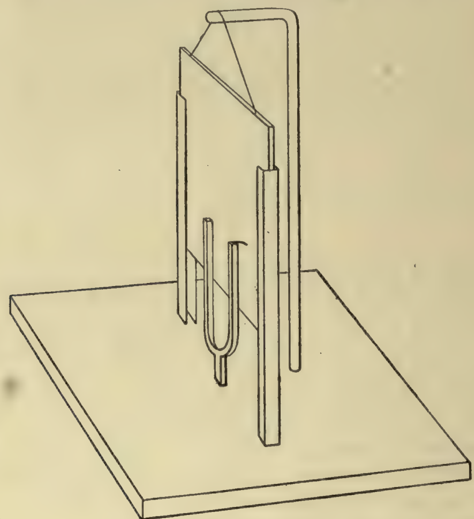


Fig. 12.

256 vibrations a second; a bristle attached to the end of the prong takes the place of the brush, and the falling body can conveniently consist of a heavy board, carrying on its face a piece of smoked glass, against which the tip of the bristle presses. A tracing will be made on the glass, which can be measured as usual; the acceleration will be found to be uniform, and its value about 981 cm. per sec. per sec. But this experiment requires

special apparatus and is not easy to carry out; so we will investigate the matter in a less direct way.

It may, or may not, be obvious that the earth attracts a body downwards with the same force whether the body is at rest or is in motion; if it be admitted that it does, then the acceleration of a body falling freely under its own weight must be uniform, for it is acted on by a constant force. If it be doubted (and the experiment in Art. 13 may suggest such a doubt), we can easily test it in this way. Take a tracing on a trolley running down a plane inclined at, say, 15° to the horizontal; the force causing the acceleration is the resolved part of the trolley's weight parallel to the plane. The acceleration is found (by the method of Ex. 15, above) to be uniform, so the component of the trolley's weight must remain the same however fast the trolley is moving. Hence the whole weight of the body (i.e. the attraction of the earth on it) must be the same whether it is in motion or at rest, since a definite fraction of it remains unchanged; hence the body if allowed to fall freely would be uniformly accelerated.

Next repeat the experiment with a different load in the trolley; the acceleration is found to be the same as before, so it is at least extremely probable from these two experiments that *all bodies, whatever their weight, have the same uniform acceleration when allowed to fall freely.*

The resistance of the air usually conceals this latter property of falling bodies, because it retards a body of small weight and large surface much more than it retards a heavy compact body. For example, if a penny and a disc of paper the size of a six-pence are released together from the same height, the penny reaches the floor long before the paper; but if the experiment is performed in a glass tube from which nearly all the air has been extracted, they fall practically side by side. This can be shown more simply if less conclusively by resting the paper on the penny, held horizontally and then dropping them; the paper falls as fast as the penny, because the latter prevents the resistance of the air from retarding the paper. Again if two weights, such as a 7 lb. and a 4 lb. weight, be dropped together from a height of five or six feet into a box of sawdust, they will arrive together as nearly as can be seen or heard; for the resistance of the air does not affect either of them seriously. So we can say that all bodies falling freely have the same acceleration. Its value is usually denoted by "g," whatever be the units in which it is expressed.

In all that follows about bodies falling freely, it must be understood that the resistance of the air is left out of account.

18. To measure the acceleration of a body falling freely. Apparatus as in Fig. 13 will be needed. It consists of a wooden bar AB about 4 ft. long, pivoted on a knife edge fixed to the wall at C; wooden studs are fixed to the wall at D and E. A thread is attached to AB at F, and after passing round E and D supports a large lead bullet H. D is so placed that H just touches the face of the bar when it hangs vertically, and the thread is made of such a length that when the bar is held by it in an inclined position as in the figure, the centre of the bullet H is just on a level with the knife edge. A strip of paper is attached to the face of the bar, so that the bullet touches it when the bar is vertical.

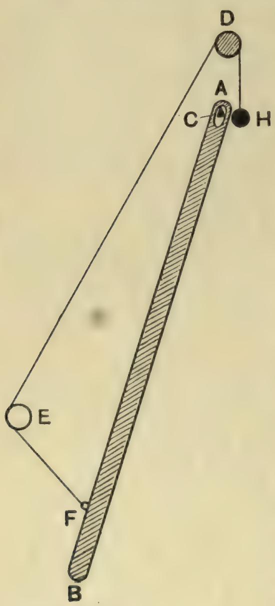


Fig. 13.

If the thread is now burned through between E and D, the bullet is released and falls freely, and simultaneously the bar swings down. When the bar reaches its vertical position, it strikes the bullet, and the latter leaves a mark on the paper; so we have a record of the distance through which the bullet falls while the bar swings down; in a particular experiment this proved to be 109 cm.

We next determine how long this bar takes to swing down to the vertical from rest in an inclined position. This is too short to measure directly, so we let the bar swing to and fro many times (say 25), measure with a watch the total time taken, and deduce the time of one quarter swing. In this case 25 swings took 47 secs., so one quarter swing takes ·47 sec.

Hence the average speed of the bullet during the first ·47 sec. is $\frac{109}{0 \cdot 47}$ or 232 cm. per sec., so the speed attained in ·47 sec. is twice 232 or 464 cm. per sec. Hence the speed attained in 1 sec. is $\frac{464}{0 \cdot 47}$ or 990 cm. per sec.; so the acceleration is **990 cm. per sec. per sec.**

It was assumed above that the time spent in one swing of the bar is the same whether the swing is long or short; during the 47 secs. spent in making 25 swings, the length (or "amplitude") of the swing decreased a great deal. This assumption can be shown to be true of any pendulum provided the swing is never large; on the truth of this depends the accuracy of clocks, for they use a pendulum as a governor, and keep the same time whether the clock is clean (when the pendulum swings vigorously) or whether it is dirty (when the pendulum swings sluggishly). This forms the best proof that the time of swing of a pendulum does not depend on the distance it swings each side of the vertical, provided this distance is small; as a matter of fact, the experiment we are describing gives a value of "g" which is only very roughly correct because of other possible errors.

The correct value for g at Greenwich is **981·17** cm. per sec. per sec., or what is equivalent, **32·191** ft. per sec. per sec. It is found to vary from place to place and with height above sea-level, but is about the same at all places on the same parallel of latitude at the same height. It is generally taken as 981 cm. per sec.

per sec., or 32·2 ft. per sec. per sec., unless another value is stated.

Ex. 16. A body falls freely from rest; what is its speed at the end of 3 secs.? what is its average speed during the first 3 secs.? how many centimetres will it fall in the first 3 secs.?

Ex. 17. Calculate the distance in feet fallen from rest, by a body falling freely, in 1, 2, 3, 4 and 5 secs. respectively.

Ex. 18. How long will it take a body falling freely from rest to attain a speed of 6 miles an hour?

Ex. 19. How long will it take a body falling freely from rest to fall 20 feet? what will then be its speed?

Call the time needed t secs. Then the average speed is $\frac{20}{t}$ ft. per sec. Hence the final speed is $2 \times \frac{20}{t}$ or $\frac{40}{t}$ ft. per sec. Now the speed attained at the end of t secs. is $32{\cdot}2 \times t$ ft. per sec. So $\frac{40}{t} = 32{\cdot}2 \times t$ or $t^2 = \frac{40}{32{\cdot}2} = 1{\cdot}243$, and $t = $ **1·11 secs.**

Its speed will be $1{\cdot}11 \times 32{\cdot}2$ or **35·74 ft. per sec.**

Ex. 20. How long will a body, falling freely from rest, take to fall 6 feet?

Ex. 21. A man jumps over a string 5 ft. 7 ins. from the ground; with what speed will he reach the ground?

19. Effect of a constant force is independent of speed. Retarding Force. We have seen that when a constant force acts on a body from rest, it continues to produce the same acceleration throughout the motion, however fast the body is moving. Hence if a constant force acts on a body already in motion, provided it acts in the same direction as the body is moving, the two general principles which we found in Arts. 14, 15 to hold for a body starting from rest under a constant force must hold for that body under the same force, even when the body is already in motion when the force begins to act; for the effect of a force cannot depend on whether the body attained its speed under the action of that force or of some other force. Hence, *a constant force produces the same uniform acceleration in a body whatever be the speed of the body when the force begins to act.*

We can now deal with cases where a body is in motion and is acted on by a constant force opposing the motion; for example, a train when the brakes are put on, or a stone thrown vertically upwards. It may be so obvious as to require no proof that the change in speed produced by the force in a second is constant whether the force is making the speed increase or decrease; but if not, it may be verified by the following experiments.

First obtain a tracing from a body known to be moving with any uniform acceleration; measure the distances between successive transits of the spring in the same direction across the centre line. It will be found that these distances increase by the same amount from one vibration to the next. (Compare Ex. 15 on page 29.)

Next start a trolley with a push *up* a sloping plane so that the friction and the component of the trolley's weight down the plane oppose the motion; obtain a tracing and measure as before the distances covered in successive equal intervals of time. These will be found to decrease by a constant amount from one vibration to the next. Hence the tracing is the same as would be got if the trolley were getting up speed under a constant force, but in this case it is traced backwards; in that case the increase of speed per second would have been constant, so in this case the decrease of speed per second was constant. This is still called "uniform acceleration" but it is said to be *negative*; it is also called "uniform retardation."

Ex. 22. A train is running at 60 miles an hour when steam is shut off. Find the distance it runs before it stops, if the constant force of friction brings it to rest in five minutes.

The retardation is constant, so its average speed is 30 miles an hour, or $\frac{1}{2}$ a mile a min.; in 5 min. at this average speed it will run **2·5 miles.**

Ex. 23. A motor car running at 20 miles an hour is pulled up by the brakes in 10 yds. Find the time taken in stopping.

Its average speed during the process is 10 miles an hour, or $\frac{10 \times 88}{60}$ ft. per sec.; the time taken to run 30 ft. at this average speed can easily be found.

Ex. 24. A motor car gets up a speed of 20 miles per hour in 10 secs., and is then stopped by the brakes in 3 secs. Compare the distances gone while accelerating and while stopping.

Ex. 25. A stone is thrown vertically upwards with a speed of 48 ft. per sec.; what is its speed and its height at the end of one sec.? How high will it rise?

It loses speed at the rate of 32·2 ft. per sec., so its speed after 1 sec. is $48 - 32{\cdot}2$ or **15·8 ft. per sec.** Its average speed is therefore $\frac{48 + 15{\cdot}8}{2}$ or 31·9 ft. per sec.; so in 1 sec. it rises **31·9 ft.** It will lose all its speed in $\frac{48}{32{\cdot}2}$ or 1·49 sec., and its average speed during that time is $\frac{48}{2}$ or 24 ft. per sec.; so its highest point is $24 \times 1{\cdot}49$ or **35·7 ft.** up.

Ex. 26. A body is thrown up with initial velocity of 200 ft. per sec. How long does it take to rise to its highest point, and to fall to the ground again? With what speed does it reach the ground?

Ex. 27. A train running at 60 miles an hour comes to rest in 200 yds. from the point at which the brakes were put on. Find the retardation caused by the brakes.

Ex. 28. A stone is thrown vertically upwards with an initial speed of 50 ft. per sec.; find its position after 2 secs.

It loses all its speed in $\frac{50}{32{\cdot}2}$ or 1·55 sec.; its average speed while rising is 25 ft. per sec.; so its greatest height is $25 \times 1{\cdot}55$ ft. or 38·8 ft. It then falls from rest, and in the ·45 sec. remaining of the 2 secs. it attains a speed of $\cdot45 \times 32{\cdot}2$ or 14·5 ft. per sec. So its average speed during this interval is 7·25 ft. per sec.; hence it falls from its highest point through $7{\cdot}25 \times \cdot45$ or 3·27 ft. So its height above the ground after 2 secs. is $38{\cdot}8 - 3{\cdot}27$ or **35·53 ft.**

Miscellaneous Exercises.

Ex. 1. A train under constant force attains a speed of 60 miles an hour in 2 mins. How much does its velocity (in ft. per sec.) increase each second? How many miles does it run in the first 2 mins.?

Ex. 2. Assuming that the force on the shot in a gun 24 ft. long is constant, find the time the shot takes to traverse the gun if the muzzle velocity is 1800 ft. per sec.

Ex. 3. A train under constant force goes 220 yds. in the first half minute from rest; how far did it go in the first quarter minute?

Ex. 4. A motor cycle travelling at a steady 50 miles an hour passes a motor car at rest, which then starts, attains a speed of 80 miles an hour in $1\frac{1}{2}$ mins. and then goes on with that speed. At what time and distance from the start will it catch the cycle?

Ex. 5. A train travelling at 45 miles an hour slips a carriage 600 yds. from a station. The carriage, with a constant retardation, is pulled up at the station. How far ahead will the train then be?

Ex. 6. A train decreases in speed from 60 miles an hour to 30 miles an hour in 2 mins.; how far does it travel in doing so?

Ex. 7. A body moves with an acceleration of 10 ft. per sec. per sec.; what is its acceleration in miles an hour per hour?

Ex. 8. The velocity of a body changes from 26 to 62 ft. per sec. while travelling 108 yds.; what is the acceleration?

Ex. 9. A body is thrown vertically upwards and rises to a height of 144 feet. With what speed will it reach the ground again?

Ex. 10. A boy standing on a bridge drops a stone into the funnel of a locomotive travelling 50 miles an hour. If he is 50 ft. above the level of the funnel, how far from the bridge is the engine when the stone is dropped?

Ex. 11. A balloon is rising at a steady speed of 30 ft. per sec. A stone is let fall from it, and reaches the ground in 17 secs. How high was the balloon when the stone was dropped? (The stone starts up at the same pace as the balloon.)

Ex. 12. Show that if a train travelling at 60 miles an hour is suddenly brought to rest by a collision, a passenger facing the engine will hit the opposite wall of the carriage as if he had fallen on it from a height of 121 feet.

Ex. 13. Two motor cars start side by side from rest. A gets up speed with an acceleration of 3 ft. per sec. per sec. for 15 secs. and then goes on with uniform velocity; B moves with an acceleration of 2·5 ft. per sec. per sec. for 20 secs. and then goes on with uniform velocity. How far ahead will A be after 20 secs.? When will they be level again?

Ex. 14. In how many seconds will a stone fall to the bottom of a pit 200 ft. deep?

Ex. 15. A body falling under its own weight is found to fall 4·025 ft. in the first half second from rest. Find the acceleration.

Ex. 16. A stone is dropped into a well and the sound of the splash is heard in 7·7 secs.; if the velocity of sound be 1120 ft. per sec., find the depth of the well.

Ex. 17. A body was acted on by a constant force which caused it to accelerate uniformly. During 5 secs. it was found to move 30 cm. and during the following 3 secs. it moved 30 cm. What was the acceleration, and with what velocity did it start at the beginning of these 8 secs.?

Ex. 18. A stone is dropped from a cliff and is seen to strike the water in 4·5 secs. What is the height of the cliff?

Ex. 19. The distances a body has moved from rest after successive intervals of $\frac{1}{8}$ sec. are observed and found to be 3, 12, 27, 48 and 75 inches. Give reasons for supposing the acceleration to be uniform, and find its value in ft. sec. units.

Ex. 20. If a train gets up speed steadily, and its average speed during the first minute is found to be 15 miles an hour, what do you mean by the statement "its velocity at the end of the first 10 secs. was 5 miles an hour"?

Ex. 21. A stone is thrown upwards with a velocity of 48 ft. per sec. Taking g as 32 ft. per sec. per sec., calculate the time from the start at which it is at a height of 32 ft. (1) going up, (2) coming down.

Ex. 22. A stone is thrown vertically upwards with a speed of 72 ft. per sec.; determine particulars of the whole subsequent motion.

Ex. 23. A shot moving at 1000 ft. per sec. penetrates 10 ins. into a target; assuming the resistance constant, find the time occupied in coming to rest.

Ex. 24. An arrow starts vertically upwards with a speed of 200 ft. per sec.; find the time when it passes a point 200 ft. above the point of projection (1) on its upward, (2) on its downward journey.

Ex. 25. A stone is thrown vertically upwards with a velocity of 96 ft. per sec.; taking g as 32 ft. per sec. per sec., find the height to which it rises, and the time from the start at which it has fallen 16 ft. from its highest point. Hence find the average speed during this time. Compare this with half the sum of the initial and final speeds; are they the same? If not, can you enlarge the law to cover such a case, by taking the sign of the velocity into account, and substituting for average speed the total change of position divided by the time?

Ex. 26. A body moves a certain distance from rest with uniform acceleration; compare the velocity at half time with that at half distance.

CHAPTER III

MASS AND WEIGHT

20. Motion of a body under different constant forces. Put a trolley on a plane sloped just enough to balance frictional resistance to motion down the plane, and attach a mass to the trolley by a thread running over a pulley at the lower end of the plane, and acting on the trolley through a spring balance. Release the trolley, and observe the force acting on it while in motion as recorded by the spring balance. Take a tracing of the motion and deduce the acceleration as follows. We know that the acceleration is uniform, so all we require is to measure the distance travelled from rest in a certain number of vibrations; we do not need to know the periodic time of the spring for this experiment. Suppose the length of say 5 wave-lengths is found to be 30 cm.; then the average speed for 5 vibrations is $\frac{30}{5}$ or 6 cm. per vibration, so the speed attained at the end of 5 vibrations is 12 cm. per vibration and the acceleration is $\frac{12}{5}$ or **2·4** cm. per vibration per vibration. Suppose that the force in this case was observed to be ·18 lb. wt.

Repeat the experiment with the same trolley but a different acting mass; suppose the force is observed to be ·27 lb. wt., and the length of 5 wave-lengths is 45 cm. This gives an acceleration of **3·6** cm. per vibration per vibration.

There is clearly a simple connection between the forces acting on the trolley and the accelerations produced; they are both increased in the same proportion (3 to 2).

21. This connection is so important that we will show it in a more accurate way, which does not depend on a measurement by a spring balance while the body is moving.

Incline the plane at a considerable slope; attach to the trolley a weight by a thread passing over a pulley at the top of the plane, thus retarding the downward motion of the trolley. Adjust the weight until the trolley runs down the plane with uniform speed when started. The forces acting on the trolley parallel to the plane are then three: (1) the component of its own weight acting down the plane, (2) the pull in the thread acting up the plane, (3) the force of friction opposing the motion, and so acting up the plane. As the speed does not alter, by Newton's First

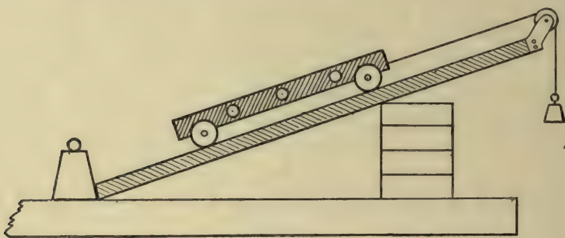


Fig. 14.

Law of Motion there is no resultant force, so the first force is just equal to the sum of the other two. Hence the pull in the string is equal to the difference between the component of the trolley's weight down the plane and the friction. If the string were to break, these two would be the only forces acting on the trolley parallel to the plane, and the resultant force causing the motion would be the difference between them; so the weight of the mass attached to the string measures this force*. (We assume that we can neglect the friction of the pulley in its bearings; with a well-made pulley, this makes no

* Since the motion is uniform, the difficulty dealt with in Art. 13 does not arise; so we run no risk of confusion here in calling the mass attached to the thread "a weight."

appreciable error.) Let the trolley run down the plane from rest, without the retarding weight being attached to it, and take a tracing of the motion; from this, determine the acceleration.

For example, suppose the trolley happened to weigh 6 lbs. and the plane happened to be inclined at 20° (there is no need to measure these), the retarding weight necessary to produce uniform speed down the plane would probably be found to be 2 lbs. wt. Suppose we find that when the trolley ran free it travelled 26·16 cm. during the first two vibrations; then its speed at the end of this time was $2 \times \frac{26 \cdot 16}{2}$ cm. per vibration, so its acceleration was $2 \times \frac{26 \cdot 16}{2 \times 2}$ cm. per vibration per vibration. Hence a force of 2 lbs. wt. produces an acceleration of 13·08 cm. per vibration per vibration in this trolley.

Next repeat the whole experiment at several different slopes of the plane. Then see whether *the accelerations are proportional to the acting forces.* For example, if the plane is sloped 13°, the retarding weight will probably be found to be 1·29 lbs. wt., and the distance travelled in the first two vibrations to be 16·88 cm. So the acceleration is 8·44 cm. per vibration per vibration. And we see that the fraction $\frac{\text{acceleration}}{\text{force}}$ in the first case was $\frac{13 \cdot 08}{2}$ or **6·54**; in the second case $\frac{8 \cdot 44}{1 \cdot 29}$ or **6·54.** (The numbers given as probable observations are calculated from the weight of the trolley, slope of the plane, etc., not taken from actual experiments, so the correspondence works out exactly; but careful experiments will show very nearly that there is a constant proportion between acceleration and acting force whenever different forces act on the same body.) Hence "*when a constant force acts on a body, the acceleration produced is proportional to the force.*"

Ex. 1. When a body is acted on by its own weight, its acceleration is 981 cm. per sec. per sec.; if this body is dragged along a perfectly smooth horizontal plane by a force equal to one-tenth of its weight, what will be its acceleration?

Ex. 2. What will be the acceleration of a 7 lb. weight lying on a horizontal sheet of ice, if a constant force of half a pound weight acts on it? (Neglect all friction.)

Ex. 3. A coal truck weighing 3 tons stands on an incline such that, if started down, it will run with uniform speed; an engine exerts a constant pull of 500 lbs. wt. on it; what will be its speed at the end of a quarter of a minute?

If the force on the truck were its own weight (3 tons wt. or 3×2240 lbs. wt.), its acceleration would be 32·2 ft. per sec. per sec.; in this case the acting force is 500 lbs. wt. instead of 3×2240 lbs. wt., so its acceleration is $\frac{500}{3 \times 2240} \times 32{\cdot}2$ ft. per sec. per sec.; so in 15 secs. its speed will be 15 times the velocity added per sec., or $\frac{15 \times 500}{3 \times 2240} \times 32{\cdot}2$ or **35·9** ft. per sec. (**24·5** miles an hour).

Ex. 4. A lift weighs half a ton; what will be the pull in the cable by which it is hung when the lift is (1) standing still, (2) dropping at uniform speed?

If the lift is allowed to fall from rest with an acceleration of 32·2 ft. per sec. per sec., what will be the pull in the supporting cable?

If the lift is allowed to fall from rest with an acceleration of 16·1 ft. per sec. per sec., what will be the pull in the supporting cable?

Since the acceleration is half of that produced by the weight of the lift, the accelerating force must be one-half the weight of the lift, so the other half (a quarter of a ton weight) must be supported by the cable.

If the lift is allowed to fall from rest with an acceleration of 1 ft. per sec. per sec., what is the pull in the supporting cable?

Here the accelerating force is $\frac{1}{32{\cdot}2}$ of the weight of the lift, so the remainder $\left(\text{i.e. } 1 - \frac{1}{32{\cdot}2}\right)$ of the weight must be supported by the cable. Hence the pull in the cable is $\frac{31{\cdot}2}{32{\cdot}2} \times 1120$ lbs. wt.

Ex. 5. Supposing that a man weighing 12 stone were standing in the lift of Ex. 4 in each case, what force would the lift exert on him?

Ex. 6. If a body rests on a smooth slope of gradient one in ten, what will be its acceleration if allowed to slide down under its own weight?

This "gradient" means a vertical rise of one unit length for a distance of 10 units measured up the slope; "smooth" means that there is no friction at all. The component of the body's weight down the slope is one-tenth of its weight (see page 31); so the acceleration will be $\frac{g}{10}$.

22. Forces needed to produce the same acceleration in different bodies. If a force produces a certain acceleration in a body, it is clear that twice this force will be needed to produce the same acceleration in two such bodies, alike in every respect, whether separate or united to form a new body; three times the force will be needed to produce the same acceleration for three such bodies; and so on. In the same way, if the body is halved, half the force will be needed to produce the same acceleration in each half; and so on.

Whatever definition we give to quantity of matter, there is clearly twice the quantity of matter in the second body as in the first, and so with the others. Hence it follows that "*the forces needed to produce the same acceleration in different bodies, made of the same material, are in the same proportion as the quantities of this material in the bodies.*"

Thus if a certain force is found to produce a certain acceleration in an iron 1 lb. mass, seven times that force will be needed to produce the same acceleration in seven iron 1 lb. masses tied together to form a single body, or in an ordinary 7 lb. mass made of iron, for it contains the same quantity of iron as seven iron 1 lb. masses tied together, according to any reasonable test we choose to apply; it weighs the same, has the same volume, would melt down into the same sized lump, etc.

Hence it is clear that in bodies made of the same material, "the quantity of this material" can be compared by their *weights.* But we could have written the last paragraph as follows, with equal correctness. "Thus if a certain force is found to produce a certain acceleration in a cubic foot of iron, seven times that force will be needed to produce the same acceleration in seven cubic feet of iron, tied together to form a single body, or in a lump of iron whose volume is seven cubic feet, for it contains the same quantity of iron as seven separate cubic feet tied together, according to any reasonable test we choose to apply; it weighs the same, has the same volume, would melt down into the same sized lump, etc." Hence we can equally well compare the quantity of material in the bodies by their *volumes.*

But if the bodies are not made of the same material, for example if one is iron and the other wood, how are we to compare the quantities of matter in them? Are we to compare their weights, or their volumes, or perhaps some other properties?

23. Quantity of matter in a body. If you buy petrol for a motor, your object is to get a quantity of the material that will run the engine for a certain number of miles; you specify the quantity you require by saying how many gallons you will take. You do not want your purchase to be bulky, except in so far as its bulk assures you that you have got the required quantity of the material. It might equally well be sold by weight, though its weight also is really only an inconvenience to you; what you want is its power of producing heat when it is burnt. Again, if you have to provide bread for a number of hungry people, you buy a certain weight of it, or lay out a certain amount of money on it, in proportion to the number of people to be fed; so you can measure the quantity of food-stuff you require by its weight or its cost. In the same way, you can measure the quantity of any one kind of material either by its bulk or its weight or its cost, though these are not the properties of the material which you are anxious to secure.

But how are we to compare the quantities of matter in a tin of petrol and a loaf of bread? It is obvious that the method to be adopted depends on the use we propose to make of them. A shillingsworth of bread contains more nourishment than a shillingsworth of petrol, it gives off less heat when burnt, and from the point of view of price, they are equal.

In dynamics we have to deal with bodies made of all kinds of material, iron, brass, wood, etc., and in order to indicate that we are regarding the quantity of matter in a body solely from the point of view of the effect of force in changing its motion, we give a special name, **Mass,** to it. So the question at the end of the last Article can be stated thus: "How are we to compare the masses of two bodies?" We will answer the question by an experiment, which shows how to find how much of one material is equal to a given mass of another material.

24. To find the quantity of one material that is equal in mass to a body made of another material. Suppose for example that the first material is wood, and the given body is an iron 4 lb. weight.

Take two trolleys, identical in all respects. Put them side by side on a table, or on two planes lying side by side and sloped just enough to overcome the frictional resistance to the trolleys' motion.

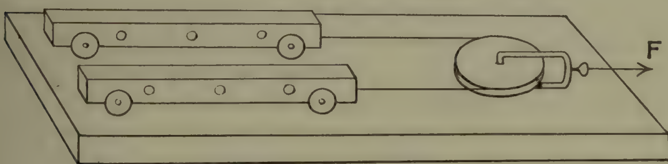


Fig. 15.

Attach one end of a thread to each trolley, and pass the loop so formed round a pulley, as in Fig. 15; apply a force to the axle of the pulley, so as to drag them along with any acceleration. The force will be equally divided between them if there is no friction at the bearings of the pulley; and as they are identical, they will move together, whatever force is applied.

Now put the given mass of iron on one of the trolleys, and a lump of wood on the other; on pulling as before, if the added masses are not equal to one another, the trolley with the greater mass will be left behind the other, because the same force does not produce so much effect on it. The amount of wood must be adjusted until they once more move together whatever force be applied to the pulley; then we know that the two bodies are equal so far as forces are concerned, that is, their masses are equal.

If now we perform a further experiment with these pieces of iron and wood, by putting them one in each pan of an ordinary balance or pair of scales we shall find that *they have the same weight.* As we have chosen the materials, amounts, etc., at random, it is clear that this will be true of all bodies which have equal mass. Hence *if two bodies made of any materials have the same weight, they have the same mass*, i.e. a given force will produce the same acceleration in each of them.

25. Distinction between mass and weight. Although this simple connection exists between mass and weight, great care must always be taken to distinguish between them. Mass is the quantity of matter in a body, weight is the force with which the earth attracts it. The mass of a body is the same wherever the body may be, whereas the weight changes from place to place; e.g. if a spring balance is accurately graduated, and a 10 lb. mass is hung on it, at Greenwich it will show that the weight of this mass is 10 lbs. wt., but if the experiment is repeated at the Equator the balance will show only 9·97 lbs. wt., and at the North Pole it would show 10·02 lbs. wt. "A pound" is not a force, but a mass; the word is often used loosely to express "the weight of a pound," but for that purpose it is much better to use the term pound-weight, or lb. wt. whenever any doubt can exist. It is allowable in such cases as "a steam-pressure of 180 lbs. per sq. in.," in which there is no doubt as to the meaning, and the custom is so firmly established that it would be pedantic to say "lbs. wt. per sq. in.," even though it would be more correct.

26. Units of Mass. It is cumbrous to deal with "the proportion between the quantities of matter in bodies" or with "the proportion between their masses," and we avoid it by adopting some unit of mass and stating the quantity of matter in a body in terms of that unit. The legal unit of mass is the quantity of matter in a certain lump of matter which is preserved as a standard.

Two such standards, of different sizes, are legalised; one is called a Pound and the other a Kilogram. In the British system of units, the Pound is the unit of mass; this system is then called the ft.-lb.-sec. system. In the centimetre-gram-second (usually written C.G.S.) system, the gram is the unit of mass, being one-thousandth of the standard Kilogram.

Copies of these standard masses can be made in any material, by making them of the same weight as the standard.

Multiples and submultiples of the unit are also made; for example, we make a 7 lb. mass by taking enough material to weigh as much as seven copies of the pound, and an ounce by making a number of pieces, each of which weigh the same, and 16 of which weigh as much as the standard pound. These are often called "weights" though *they are not standards of weight at all*; as a matter of fact, the weight of one of these standards changes as it is moved from place to place on the earth. But if we determine the weight of a body by means of a balance and these "weights," we know the mass of the body, in pounds or grams; if it weighs m lbs., we know that the force needed to produce a certain acceleration in the body is m times the force which will produce the same acceleration in a body of mass 1 lb.

27. Kinetic units of Force. Up till now we have measured forces as in Statics, by the attraction of the earth on a standard mass or by the extension of the spring of a spring balance. As this method is not a convenient one in many dynamical problems, we will now explain another way of measuring forces.

As usual we need a unit in terms of which we may state the magnitude of the forces we are using, and the most convenient is one which depends directly and solely on the units of mass, length and time; such a unit is usually termed an **Absolute Unit. The absolute unit of force is "that force which acting on a unit mass produces in it unit acceleration."**

If we are using the British, or Foot-Pound-Second, units, the British absolute unit of force will be "that force which acting on a mass of 1 lb. produces in it an acceleration of 1 ft. per sec. per sec." It is called a **Poundal.**

The C.G.S. unit of force is "that force which acting on a mass of 1 grm. produces in it an acceleration of 1 cm. per sec. per sec." It is called a **Dyne.**

The Poundal and the Dyne are called *Kinetic* units of force, because they depend on the movement which they produce in bodies.

Suppose we wish to express the value of any given force in Poundals. We apply the force to the standard mass of 1 lb. and measure the acceleration it produces; call this a ft. per sec. per sec. Now we know from experiment (Art. 21), that when different forces act on the same body, the accelerations are proportional to the forces. Since by definition one Poundal would produce in this body an acceleration of 1 ft. per sec. per sec., the force to be measured must be a poundals.

If the force is large, it would be inconvenient to test it on so small a mass as 1 lb., for it would produce an acceleration so large as to be hard to measure. Suppose we apply the force to a mass of m lbs., and find that it produces an acceleration of a ft. per sec. per sec. We have seen (Art. 22) that this force must be m times as great as that needed to produce the same acceleration in a mass of 1 lb., and we have just seen that this latter force is a poundals. Hence the force we are measuring is $m \times a$ poundals.

Similar reasoning shows that the force needed to produce an acceleration of a cm. per sec. per sec. in a mass of m grms. is ma dynes.

If we denote the force by F poundals or dynes according as we are measuring m and a in British or C.G.S. units, then we have for both systems the equation

$$F = m . a.$$

Ex. 7. A body of mass 7 lbs. moves under a constant force, and at the end of 1·2 sec. from rest it has covered 18 inches; what is the force?

The average speed is $\frac{18}{1 \cdot 2}$ ins. per sec. or $\frac{18}{1 \cdot 2 \times 12}$ ft. per sec.; so its speed at the end of the 1·2 sec. is $\frac{2 \times 18}{1 \cdot 2 \times 12}$ ft. per sec. So its acceleration is $\frac{2 \times 18}{1 \cdot 2 \times 12 \times 1 \cdot 2}$ ft. per sec. per sec., or 2·08 ft. per sec. Hence using the equation $F = ma$ the force on it is $7 \times 2 \cdot 08$ or **14·56 poundals.**

Ex. 8. A body of mass 3000 grms. moves under a constant force, and at the end of ·9 sec. it has moved 32·0 cm. from rest; calculate the force in dynes.

Ex. 9. A body weighing half a ton is acted on by a constant force of 600 poundals; calculate its acceleration and hence its speed at the end of a minute.

28. Relation between gravitational and kinetic units of force. Since the kinetic units of force depend only on the units of mass, length and time, and do not depend on the position of the place where they are used, they are called "absolute" units of force. The unit of force commonly used by engineers, the weight of one pound mass, is called a "gravitational" unit. The magnitude of this force varies with the position of the pound mass, being greater in high latitudes than near the equator, and at sea level than at the top of a mountain. Hence if precision of definition is important it is necessary to name a definite spot where the attraction of the earth on the standard pound mass is to be measured, and Greenwich is usually adopted. Since this force (the weight of a pound at Greenwich) acting on a body whose mass is one pound is found to produce in it an acceleration of 32·191 ft. per sec. per sec., the force is 32·191 poundals. So one pound weighs 32·191 poundals at Greenwich, and a poundal is roughly the weight of half an ounce.

The weight of one pound mass at any other place is not accurately "1 lb. wt." as defined above, but its value can be calculated when the value of g at that place is known. For all easily attainable places the change in the weight of one pound is small, and the error produced by calling it "1 lb. wt." can often be neglected; e.g. at the equator it is only an error of $\cdot 3\,^\circ/_\circ$.

"The weight of one pound at Greenwich" is universally adopted by engineers as unit force, not only in Statics, but also in cases where there is a change of velocity (e.g. the draw-bar pull which accelerates a train, the force needed to overcome the inertia of a piston, the pressure of steam on the blades of a turbine, etc. are expressed in lbs. wt.). In such cases it is much safer for beginners to work out the problem in poundals, and reduce the result to lbs. wt., if it be desired to express it in that way.

It may be noted here that the weight of a body whose mass is M grms. is $981 \times M$ dynes, since this force if unresisted will give the body an acceleration of 981 cm. per sec. per sec.

Ex. 10. Calculate the constant force (1) in poundals, (2) in lbs. wt., needed to get up a speed of 30 miles an hour in 20 secs. in a motor car weighing 1 ton. (Half of this is the force which the tyre of each driving wheel exerts on the road.)

Ex. 11. A motor car weighing 1 ton and running at 20 miles an hour is pulled up by its brakes in 10 yards; calculate in poundals and in lbs. wt. the force exerted by the tyres.

Its average speed is 10 miles an hour, or $\frac{10}{60} \times 88$ ft. per sec. Hence the time occupied in stopping is $\frac{10 \times 3 \times 60}{10 \times 88}$ secs. So its acceleration is $\frac{20 \times 88 \times 10 \times 88}{60 \times 10 \times 3 \times 60}$ ft. per sec. per sec., and the force of the tyres is $\frac{2240 \times 20 \times 88 \times 10 \times 88}{60 \times 10 \times 3 \times 60}$ poundals or $\frac{2240 \times 20 \times 88 \times 88}{32 \cdot 2 \times 60 \times 3 \times 60}$ lbs. wt.

Ex. 12. A lift weighs 5 cwt.; find in poundals and lbs. wt. the pull in the cable supporting it (1) when it is at rest, (2) while it is moving upwards at constant speed, (3) while it moves upwards with uniform acceleration of 6 ft. per sec. per sec.

In cases (1) and (2) the pull of the cable balances the weight of the lift, so it is 5×112 or 560 lbs. wt., or $560 \times 32 \cdot 2$ poundals. In case (3) the resultant force on the lift is 560×6 poundals, so the pull in the cable is $560 \times 32 \cdot 2 + 560 \times 6$ poundals, or $\frac{560 \times 38 \cdot 2}{32 \cdot 2}$ lbs. wt.

Ex. 13. A man weighing 12 stone stands in a lift, which is descending with an acceleration of 2 ft. per sec. per sec.; what pressure (in poundals and in lbs. wt.) does he exert on the floor of the lift?

Miscellaneous Exercises.

Throughout these exercises, friction is to be disregarded, and the value of "g" may be taken as 32 *in British, and* 980 *in* C.G.S. *units.*

Ex. 1. Find the force in poundals needed to produce an acceleration of 20 ft. per sec. per sec. in a mass of 20 lbs.

Ex. 2. Find the force in dynes needed to produce an acceleration of 100 cm. per sec. per sec. in a mass of 1 kilogram.

Ex. 3. Find the velocity of a mass of 56 lbs. after a force of 100 poundals has acted on it for 3 seconds.

Ex. 4. A body acted on by a constant force of 60 poundals for 2 secs. from rest has a velocity of 88 ft. per sec.; what is its mass?

Ex. 5. A train of 100 tons running at 60 miles an hour is brought to rest by friction in 30 seconds; what is the effective force?

Ex. 6. Find the force in poundals needed to produce an acceleration of 32 ft. per sec. per sec. in a mass of 3 lbs.

Ex. 7. Find the vertical force in poundals needed to retard a body of mass 3 lbs., so that it may fall with an acceleration of 16 ft. per sec. per sec.

Ex. 8. A train of mass 100 tons stands on an incline of 1 in 100; find in lbs. wt. the component of its weight down the plane. What is this force in poundals? An engine exerts a constant draw-bar pull of 4 tons wt. on the above train down the incline; find this force in poundals; hence find the acceleration.

Ex. 9. Find the acceleration when the engine in Ex. 8 pulls the train up the incline.

Ex. 10. A train of mass 100 tons running at 60 miles an hour is brought to rest by the brakes in 10 secs. on the level. Find the braking force in lbs. wt. Find also the time and distance in which it would be brought to rest by the same braking force from that speed when running (1) up, and (2) down, an incline of 1 in 50.

Ex. 11. A crane pulls up a mass of 200 lbs. (1) with uniform speed, (2) with an acceleration of 3 ft. per sec. per sec.; and then lets it down with an acceleration of 4 ft. per sec. per sec. Find in poundals and lbs. wt. the pull in the chain in each case.

Ex. 12. A locomotive engine of mass 50 tons attached to a truck of mass 25 tons moves with an acceleration of 2 ft. per sec. per sec. Find (1) the total force exerted by the engine on the rails, (2) the force exerted on the truck.

Ex. 13. Compare the masses of two bodies, which are found to require forces of 1 lb. wt. and 1000 dynes respectively, to give them the same acceleration. (1 inch = 2·54 cm., 1 lb. = 454 grms.)

Ex. 14. Find the number of dynes in 1 poundal. (1 lb. = 454 grms., 1 inch = 2·54 cm.)

Ex. 15. A train runs freely down an incline of 1 in 80. Find its speed in miles per hour after running half a mile.

Ex. 16. A rifle has a barrel 30 ins. long, and fires a bullet weighing half an ounce, with a muzzle velocity of 2000 ft. per sec. Assuming that the pressure of the gas in the barrel is constant while the bullet is in it, find the acceleration of the bullet and the force on it.

CHAPTER IV

FORCE INCLINED TO DIRECTION OF MOTION

29. Action of a force at an angle with the direction of a body's motion. We have hitherto considered the effect of a force which acts constantly in the same direction as the body moves. There are many cases in which the force acts at an angle with that direction, making the body move out of its straight path; for example, if we watch the flight of a stone thrown horizontally we see that its path curves downwards till the stone strikes the ground, because it is pulled down out of its straight path by the attraction of the earth.

It is not obvious at first sight how such a force will affect the velocity of a body; the example of the flight of a stone suggests that a force, acting at an angle with the velocity which a body has at the moment, changes that velocity into one more in the direction of the force; but it is not clear how rapidly this change of direction takes place, nor whether the speed of the body is altered. We will proceed to investigate these questions, and must begin by getting clear ideas about the motion of a body whose velocity is continually changing in magnitude and direction. In order to simplify the matter we will at first consider only the displacement which a body undergoes, and not attend to the time occupied in the displacement.

30. Composition of Displacements. Suppose that a body is moved a certain distance, say 3 feet, horizontally and then a certain distance, say 4 feet, vertically upwards. This clearly comes to the same in the end as if it had been moved a certain distance upwards at a certain angle with the vertical. We can represent the displacement by a diagram drawn to scale, as in Fig. 16, where AB, BC represent the two separate actual displacements, forming two sides of a triangle taken in order, and AC is what we may call the "resultant displacement," which is the third side of the triangle. The body was not actually displaced along the line AC, but the final result is the same as if it had been.

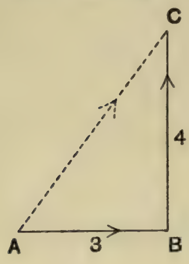


Fig. 16.

Ex. 1. A man walks 3 miles N.E., then 2 miles N.W., then 4 miles due S. Find, by drawing these displacements to scale, the magnitude and direction of his resultant displacement.

Ex. 2. A man walks 15 ft. across a lift, which then ascends 40 ft.; find by a diagram to scale the magnitude and direction of his resultant displacement.

31. Resolution of Displacements. If a body receives a displacement from one point to another, this is equivalent to two displacements, one of which may be anything we choose provided that the other brings the body to the same finishing point as before.

If AB represents (in magnitude and direction) the actual displacement of the body, then two displacements represented by AC_1, C_1B, or two represented by AC_2, C_2B, would produce the same result. So any displacement represented by a straight line is equivalent to two which are represented by two straight lines forming any triangle with the first, the sides of the triangle being taken in order (e.g. in Fig. 17 the two straight lines must

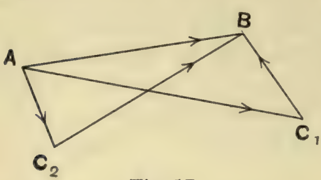


Fig. 17.

be AC_2 and C_2B, not AC_2, BC_2; for a displacement represented by BC_2 would move the body from, not towards, B in the second movement). This process of finding two displacements which are together equivalent to one is called "resolving" that displacement into two "components"; it is usually most convenient to resolve a displacement into two others at right angles to one another, and unless the contrary is stated, the components are assumed to be at right angles to one another.

Inspection of Fig. 18, in which AC_2 is parallel to C_1B, and C_2B is parallel to AC_1, shows that when a displacement is resolved into two components we can take either component as the first displacement; the rule about taking the sides of the triangle in order shows the "sense" of each displacement (e.g. in Fig. 18 it is AC_2, not C_2A, and C_1B, not BC_1).

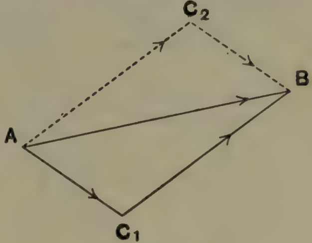


Fig. 18.

For example, a displacement of 5 miles N.N.E. is equivalent to, or may be resolved into, one of 1·91 miles due E. and 4·62 miles due N.; these values of the components may be found by a scale drawing or by trigonometry (5 sin 22·5° and 5 cos 22·5°). We can take these components in either order, going 4·62 miles N. and then 1·91 miles E., or first E., then N., as we please.

Ex. 3. A man walks 4 miles N.W.; resolve his displacement into two, one in the N. and S. line, the other E. and W.

Ex. 4. A man walks 4 miles N.W. and then 5 miles N.E.; find graphically the components of his resultant displacement in the N. and E. directions.

Ex. 5. A man walks 1 mile S.W., then 2 miles due S., then 4 miles E., then 3 miles N.E.; how far is he E. of his starting point?

32. Simultaneous Displacements. Suppose that a body receives displacements in two or more directions simultaneously instead of one after the other (for example, if a man walks across a lift while it is ascending); then the final result is the same in both cases. So what we have found for the composition and resolution of successive displacements holds good for simultaneous displacements.

If in addition we know that the speed in each of the displacements is uniform, we can easily determine the actual path of the body during the time it is moving.

For example, suppose that a man walks at a steady pace of 3 miles an hour across a lift 11 ft. wide while it ascends at a speed of 8 ft. per sec. 3 miles an hour is $\frac{3}{60} \times 88$ ft. per sec. or 4·4 ft. per sec., so he will cross the lift in $\frac{11}{4 \cdot 4}$ or $2\frac{1}{2}$ secs. In this time the lift will ascend $2\frac{1}{2} \times 8$ or 20 ft. So his total displacement at the end of $2\frac{1}{2}$ secs. is represented by AC (Fig. 19) if AB represents 11 ft. and BC represents 20 ft. Note that AB and BC do not now represent the lines along which he actually moves, as they did when the displacements were made separately. If we calculate his actual position at any intermediate time we shall find that his displacement up to that time is in the direction of AC; for example, at the end of 1 sec. he will have moved 4·4 ft. across the lift, and it will have ascended 8 ft.; if we mark these lengths AM, MP on the diagram we find that his displacement AP lies along AC, and as M moves with uniform speed along AB, P clearly moves with uniform speed along AC, since it is always vertically above M; so it is clear that the actual motion of the

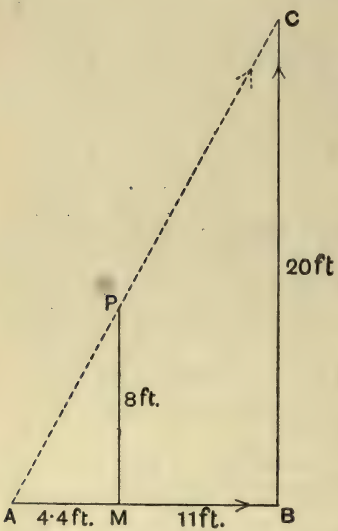
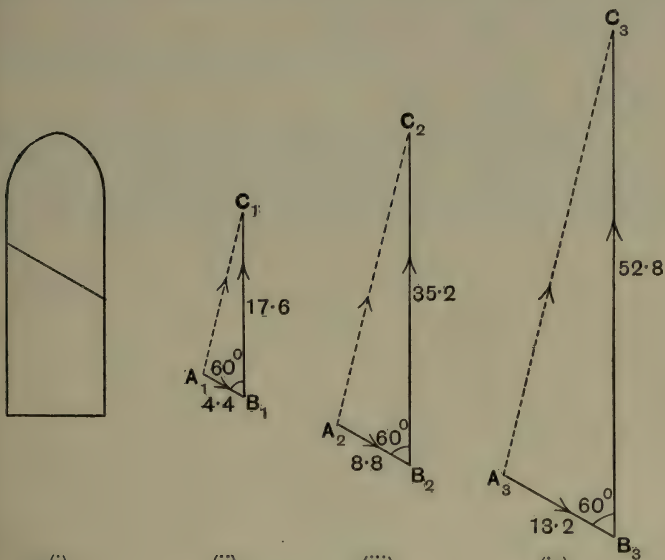


Fig. 19.

man is a uniform speed along a straight line from his starting to his finishing point.

As another example, consider the case of a man walking diagonally across a ship which is itself in motion; suppose he walks at 3 miles an hour in a direction inclined at 60° to the



(i) (ii) (iii) (iv)

Fig. 20.

centre line of the ship towards the stern; suppose the ship is 90 ft. broad and runs at 12 miles an hour. Then his path across the deck will be as in Fig. 20 (i) and his actual displacement at the end of a second will be A_1C_1 in Fig. 20 (ii), the resultant of two represented by A_1B_1 (4·4 ft.) and B_1C_1 (17·6 ft.); A_1C_1 will be found to represent 15·9 ft.; at the end of 2 secs. it will be A_2C_2 in Fig. 20 (iii), which is drawn to the same scale as

Fig. 20 (ii), where A_2B_2 represents 8·8 ft. and B_2C_2 35·2 ft. and A_2C_2 will be found to represent 31·8 ft.; at the end of 3 secs. it will be A_3C_3 in Fig. 20 (iv), and so on. It is clear that these triangles differ only in size, and the sides increase uniformly with the time; so the actual displacement of the man is constant in direction and he moves in that direction with constant speed.

***33.** It is easy to give a general proof of this result, if it be desired.

Suppose that a body moves with uniform velocity (v_1) with respect to another body which itself is moving with uniform velocity (v_2), and let AB, BC represent the respective displacements in a certain time, T secs. Then $AB = v_1 T$ and $BC = v_2 T$.

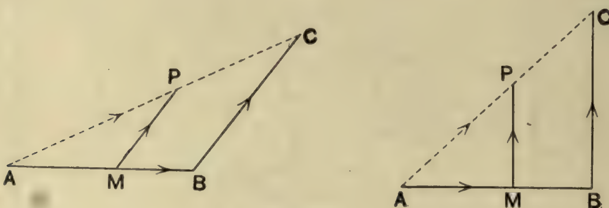


Fig. 21.

At any other time t secs. from the start the displacement will be $AM = v_1 t$ and $MP = v_2 t$; hence $\frac{PM}{AM} = \frac{v_2 t}{v_1 t} = \frac{v_2}{v_1}$, and similarly $\frac{CB}{AB} = \frac{v_2}{v_1}$, so $\frac{PM}{AM} = \frac{CB}{AB}$. So the triangles APM, ACB are similar to one another, having the angles A the same; so P lies on AC. Again $\frac{AP}{AM} = \frac{AC}{AB}$. But $AM = v_1 t$, so $AP = \frac{AC}{AB} \times v_1 t$, and AP is proportional to t; i.e. the actual displacement of the body increases uniformly with the time, or the body actually moves with uniform velocity.

34. Composition and Resolution of Uniform Velocities. Since a uniform velocity is represented by the displacement which takes place in unit time, we can substitute "uniform velocity" for "displacement" in what we have found to be true of the latter. So we have, **if a body has simultaneously two uniform velocities, represented by two sides of a triangle taken in order, its actual velocity is represented by the third side.**

For example, if a ship is steaming through the water at 7 miles an hour, steering due N., and meanwhile the tide is carrying her due W. at a speed of 3 miles an hour, then the ship has simultaneously two uniform velocities, represented in Fig. 22 (i) by AB and BC, and her actual velocity is represented

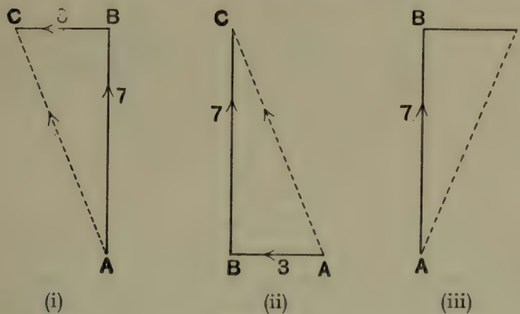


Fig. 22.

by AC. Note that neither A nor B represents the ship, nor does she move along the actual lines AB, BC in an hour; these lines are *parallel to*, *proportional to* and *in the same sense as*, the displacements which take place in an hour, and represent the displacements or velocities to that extent. They can equally well be placed anywhere in the diagram, but in order to find the resultant velocity they must be drawn as two sides of a triangle taken in order; so they can either be as in Fig. 22 (i), or as in Fig. 22 (ii), but not as in Fig. 22 (iii).

Ex. 6. A motor boat crosses an estuary half a mile wide, going at a speed of 8 miles an hour through the water and steering always straight for the opposite shore; a tide of 3 miles an hour carries her down stream. Find her actual velocity by a scale drawing, and find how far she moves (with respect to the land) in crossing, how far she is carried down stream, and the time she takes. Is this last the same as if there was no tide?

Ex. 7. The motor boat of the last example wishes to reach a point on the opposite shore exactly opposite her starting point; in what direction must she steer?

She starts from A and wishes to reach B, so her resultant velocity must be along AB. One of the components is 3 miles an hour parallel to the shore; represent it by AC, representing 3 miles on some scale.

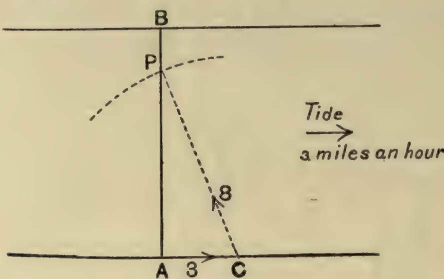


Fig. 23.

We know that the other component is 8 miles an hour; we do not know its direction, but it may be any line drawn from C, 8 miles long on the same scale. So its end lies on a circle with centre C, radius 8 on this scale. In order that the resultant velocity may be along AB, the end of this second component must be on AB, or AB produced, so it will be P at the intersection of AB and the circle. The direction CP will then be the required direction.

Ex. 8. Find how long the motor boat in the last example takes to cross the estuary. How far does she move (1) with respect to the land, (2) through the water, in doing so?

35. Composition and Resolution of Velocities which are changing. Consider as an example of a body whose velocity is changing both in magnitude and direction, a bullet fired from a gun with a muzzle velocity of 2000 ft. per sec. at an elevation of 15° above the horizontal. Directly it has left the gun the resistance of the air begins to reduce its speed, and its weight begins to deflect it from its straight path; but *at the instant* of leaving the gun we know what its velocity is, i.e. the *uniform* velocity with which it would proceed if these forces ceased to act, so that it would move 2000 ft. in the first second at an elevation of 15°. As this is a uniform velocity, we can deal with it as we have done with other uniform velocities, and resolve it into two components in the same way. It is convenient to take these components in the vertical and horizontal directions. Then a scale drawing (or calculation by trigono-

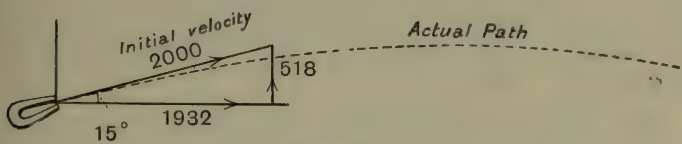


Fig. 24.

metry) shows that the horizontal component is 1932 ft. per sec., and the vertical component is 518 ft. per sec. (The horizontal component is $2000 \times \cos 15°$ and the vertical component is $2000 \times \sin 15°$.)

It may help in realising the meaning of these ideas to imagine that the sun is vertically overhead; then the speed of the shadow of the shot on the ground is the horizontal component of the velocity (or "the horizontal velocity" as it is often written). This shadow starts with a speed of 1932 ft. per sec., but we have not yet found out whether it actually covers 1932 ft. in the first second; all we know is that it would do so if the forces on the shot ceased to act. Similarly the vertical component can be thought of as the speed of the shadow of the shot thrown on a vertical wall behind the gun by a sun just setting in the direction in which the shot is fired; if the weight and the resistance of the air ceased to act this shadow would rise 518 feet in the first second.

Hence we see that if a body's velocity is changing, we can resolve or compound its value at any instant as though it were constant.

Ex. 9. If a man starts to move at 2 miles an hour across a railway carriage which at that instant is moving at a speed of 8 miles an hour, what is his actual speed?

By compounding the two velocities we find that his actual velocity at the instant is 8·25 miles an hour at an inclination of 14° to the rails.

Ex. 10. Rain is falling vertically at a speed of 120 ft. per sec.; find the inclination to the vertical at which the drops, as seen by a passenger in a train running at 60 miles an hour, seem to fall.

Ex. 11. Rain-drops are moving through the air at a speed of 70 miles an hour, and are moving horizontally with the wind at a speed of 20 miles an hour; find the angle at which an umbrella should be held by a man standing still.

36. Change in Velocity. If a body always moves in one straight line, and is moving faster at one time than another, it is easy to find the change in its velocity. If at both times it is moving in the same direction along the line, it is the difference between the speeds at the two instants, but if in the interval between them the motion has been reversed the change in velocity is clearly the sum of the speeds. Since velocity, unlike speed, is called positive when in one direction along a line and negative in the other, the change in both cases is "the difference of the velocities." For example if at one instant the body has a speed of 3 ft. per sec. along a straight line, in the direction from A to B which we will call the positive direction, its velocity is $+3$; if at a later time its speed is 5 ft. per sec. in the same direction, its velocity is then $+5$ and the change in the velocity is $5-3$ or 2 ft. per sec. But if at the later time its speed is 5 ft. per sec. in the direction BA, its velocity is then -5, and the change is $-5-3$, or -8 ft. per sec. The negative sign implies that there has been a reduction in the velocity in the positive, or A to B, direction; in the first instance the positive sign showed that the velocity had increased.

If a body alters its direction of motion, the determination of its change of velocity is not so simple. Take as an example the case of a man walking into a lift which starts upwards before he reaches the opposite side; his velocity is at first horizontal, and later is obliquely upwards; as before, if AB represents his velocity along the floor, and BC the velocity of the lift when it has got up speed, AC will represent his resultant velocity. The change in his total velocity is clearly produced by and equal to the actual velocity of the lift; so BC represents the *change*

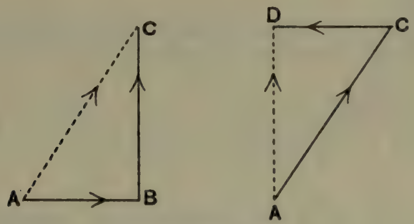


Fig. 25.

which has taken place in his velocity, altering it from AB to AC. Again, when he stops walking, if the lift continues to rise, his velocity changes from AC to AD, and CD represents the change in his velocity.

Hence in general if we wish to find the change in velocity of a body, we have only to find that velocity which compounded with the first gives the second velocity. We can do so by drawing from one point lines to represent the two velocities of the body, then the line joining their ends represents on the same scale the change in the velocity, and the sense of this change of velocity is from the end of the line representing the former to the end of that representing the latter velocity.

Ex. 12. At a certain instant a cricket ball is rising at an angle of 45° to the horizontal with a speed of 45·4 ft. per sec.; two seconds later it is falling at an angle of 45° to the horizontal with the same speed. What is its change of velocity?

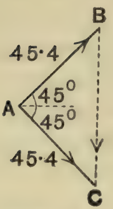


Fig. 26.

In Fig. 26 AB represents its former and AC its latter velocity, so the change is represented by BC. By measurement or calculation BC will be found to represent a velocity of 64·4 ft. per sec. vertically.

Note that although the speed is the same, the velocity has changed.

Ex. 13. At a certain instant a cricket ball is rising at an angle of 20° to the horizontal with a speed of 60 ft. per sec.; one second later it is falling at an angle of 11° 42′ to the horizontal with a speed of 57·5 ft. per sec. Determine by a scale drawing its change of velocity.

Ex. 14. A motor car runs along a road at 15 miles an hour, then turns into another road at right angles and goes at 20 miles an hour; determine its change of velocity.

Ex. 15. A man runs with uniform speed of 15 miles an hour round a circular track of 4 laps to the mile; determine his change of velocity in 10 secs.

37. Motion of a body projected horizontally. If you are sitting in a train which is running at constant speed, and drop something, would your experience lead you to expect that it would move in the same way as if you had been sitting in a room when you dropped it? would you expect it to hit the floor vertically below where it started, or perhaps nearer to the engine? Everyone assumes that a body in motion behaves in the same way in a railway carriage whether the carriage is at rest or in uniform motion, and acts on that assumption when trying to catch it when it falls or is thrown to them. So it is fair to take this experience as proof that if a body has a horizontal velocity when it begins to fall freely under the earth's attraction, that attraction will not alter the horizontal velocity. If it did so, the body would not *appear* to fall vertically to the passenger who himself has all the time the same horizontal velocity, and it would hit some other point of the floor of the carriage than that immediately below it when it started to fall. As a matter of fact of course since the carriage is moving it does not really move vertically, and would not appear to do so to anyone standing on a platform.

But it would not be fair to assume from this experience that the action of the earth on the body is not affected by the horizontal velocity; we should not be likely to perceive the difference if the body only fell say 3·8 ft. in the first quarter second when the train was running at 60 miles an hour, whereas we know it would fall 4 feet if the train were at rest, such

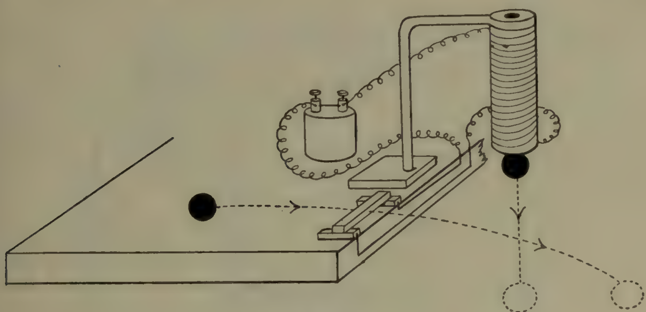


Fig. 27.

difference in speed being imperceptible without careful measurement. However a simple experiment will show whether or not the acceleration of a body falling freely is affected by any horizontal velocity which the body has at the start.

If we roll a metal ball along a table, and if at the instant when it reaches the edge we release another ball from the same height so that the latter falls straight downwards, we shall hear them hit the floor at the same moment, whatever horizontal velocity the first ball had when it left the edge of the table. As it is not easy to release the second ball at the exact instant when the other leaves the table, it is better to make the rolling ball release the other at that instant by working a trigger; this trigger may most conveniently break the circuit of an electro-magnet which supports the ball that is to fall vertically, as in Fig. 27. The paths of the two balls are indicated by dotted lines.

The result of this experiment shows that the action of a constant force on a body produces the same effect whatever may be the velocity of the body; and we have seen that the velocity of a body at right angles to the force that acts on the body is unaffected by that force. We will now show the latter by an experiment which does not require to be made in a railway train.

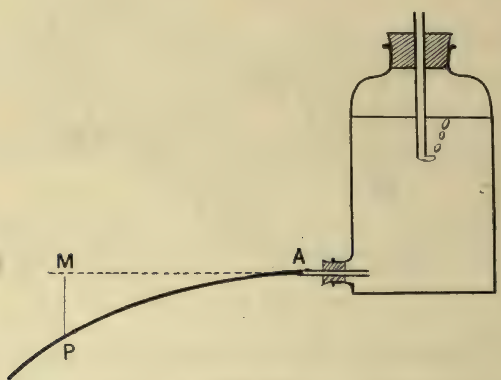


Fig. 28.

38. Path of a water jet starting horizontally. A convenient way of observing the path of a body projected with an initial velocity under the earth's attraction is to use a jet of water, which is equivalent to a succession of projectiles; its shadow, thrown by a distant lamp at the same level, can easily be traced on a piece of paper held close to the jet. In order to keep the initial velocity constant for some time, an arrangement called a Marriotte's bottle should be used, as shown in Fig. 28.

If we take any point P on the path, by measuring PM we can deduce the time since the particle of water left A until it reached P; for it is the same time as a particle falling freely from rest would drop through the vertical height PM, the difference of level between A and P. If we now measure the distance AM we can deduce the average horizontal component of the velocity of the body during this time. If we repeat this for different positions of P we shall find that the horizontal component of the velocity is constant (neglecting the small reduction caused by air resistance).

Hence, a force produces no effect on the motion of a body at right angles to the force.

Ex. 16. A stone is thrown horizontally with a speed of 60 ft. per sec. from the top of a cliff 100 ft. high; where will it strike the sea?

Call the time before it does so, t secs. In t secs. a body falling freely from rest would acquire a velocity of $32{\cdot}2 \times t$ ft. per sec., so its average speed would be $\frac{1}{2} \times 32{\cdot}2 \times t$ or $16{\cdot}1 \times t$ ft. per sec. In t secs. at this speed it would cover $16{\cdot}1 \times t \times t$ or $16{\cdot}1t^2$ ft. Now such a body and the stone we are considering would take the same time to reach sea-level, hence $100 = 16{\cdot}1 \times t^2$ and $t = 2{\cdot}49$ secs. As its horizontal velocity of 60 ft. per sec. remains unchanged during the flight, it will travel horizontally a distance of $60 \times 2{\cdot}49$ or 149·4 ft.; so it will strike the sea **149·4** ft. from the cliff.

Ex. 17. Plot the path of a water-jet issuing horizontally with a speed of 5 ft. per sec., on a scale of 1 inch to 1 ft.

Calculate the vertical heights through which it will fall in ·1, ·2, ·3 etc. secs.; the horizontal distances it will have moved will then be ·5, 1·0, 1·5 etc. ft., and the actual positions can be marked on the diagram and a smooth curve drawn through them.

Ex. 18. A stone is thrown horizontally with a speed of 50 ft. per sec.; find its velocity in magnitude and direction after 2 secs.

Compound a vertical velocity of $2g$, or 64·4, ft. per sec. with a horizontal velocity of 50 ft. per sec.

Ex. 19. A stone is thrown horizontally with a speed of 60 ft. per sec. from a cliff, and is seen to strike the water 3 secs. afterwards; find the angle at which it does so.

39. Inclined projection. Hitherto we have assumed that the body, which falls freely and at the same time has a uniform horizontal velocity, was originally projected horizontally, but this was only done for the sake of simplifying the idea; the effect of the earth on a moving body cannot depend on how the motion started. So we can apply our results to a body which has started its flight at any angle.

Suppose then that we have a body projected at an elevation of 60° above the horizontal, with a velocity of 100 ft. per sec. Resolve this velocity into a horizontal and a vertical component; they will be respectively 100 cos 60° or 50 and 100 sin 60° or 86·6 ft. per sec. Then the body will retain this horizontal velocity of 50 ft. per sec. throughout the flight; but the vertical velocity will change exactly as with a body projected vertically upwards with a velocity of 86·6 ft. per sec. From these two facts we can deduce anything we wish to know about the flight of the body.

For example, to find how long it will be before the body comes down to the same level as its starting point. It loses 32·2 ft. per sec. of its vertical velocity each second, so it will reach its highest point in $\frac{86 \cdot 6}{32 \cdot 2}$ secs.; a body projected vertically upwards takes the same time to fall to its starting point from its highest point as it did to rise (see Ex. 11, chap. II), so the total time of flight of this body is $2 \times \frac{86 \cdot 6}{32 \cdot 2}$ or 5·38 secs. In that time, since it has a constant horizontal velocity of 50 ft. per sec., it will travel horizontally through $50 \times 5 \cdot 38$ or 269 ft. So if projected from the ground level, it will strike the earth at a distance of 269 ft.

The distance travelled horizontally before the projectile returns to the same level as its starting point is usually called the "range on the horizontal plane."

Ex. 20. A shot is fired at an elevation of 15° with a muzzle velocity of 2000 ft. per sec.; neglecting the resistance of the air, find its range on the horizontal plane.

Ex. 21. Find how high the shot in Ex. 20 will rise.

Ex. 22. A cricket ball is thrown with a speed of 60 ft. per sec.; determine its horizontal range if the angle to the horizontal at which it is thrown is 20°, 30°, 40°, 50°, 60° or 70°. Make a diagram of these ranges, taking angles of elevation as abscissae and ranges as ordinates. What angle of elevation gives the greatest range for a given initial speed? What other angle of elevation gives the same range as one of 30°?

40. Suppose we wish to determine the position of a projectile at any instant of its flight; we can do so by finding its vertical and horizontal displacements at that instant. For example, if a cricket ball is thrown with a speed of 50 ft. per sec. at an angle of 60°, its initial horizontal velocity will be 50 cos 60° or 25 ft. per sec., and its vertical velocity will be

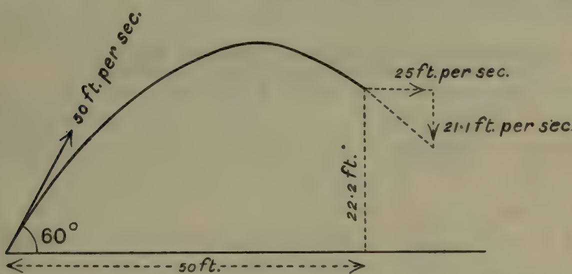


Fig. 29.

50 sin 60° or 43·3 ft. per sec. At the end of 2 secs. it will have moved 50 *ft. horizontally.* It loses 32·2 ft. per sec. of its vertical velocity each second, so it will stop rising after $\frac{43 \cdot 3}{32 \cdot 2}$ or 1·345 sec. Its average vertical speed during that time is $\frac{43 \cdot 3 + 0}{2}$ or 21·65 ft. per sec., so it will rise to a height of 1·345 × 21·65 or 29·12 ft. In the remaining ·655 sec. it will acquire a vertical velocity of ·655 × 32·2 or 21·1 ft. per sec., so it will fall through $\frac{0 + 21 \cdot 1}{2} \times \cdot 655$ ft., or 6·91 ft. Hence it is then 29·12 – 6·91 ft. or 22·21 *ft. above its starting level.* These give us its actual position after 2 secs. of flight.

If we wish to know its velocity then, we must compound its horizontal and vertical velocities at that instant; we have found that these are respectively 25 ft. per sec. and 21·1 ft. per sec. (downwards); when compounded they give an actual velocity of 32·7 ft. per sec. at an angle below the horizontal of $39\frac{1}{2}°$ nearly. These results are shown in Fig. 29.

Ex. 23. A stone is thrown with a velocity of 80 ft. per sec. at an elevation of 45° above the horizontal; find its position at the end of the first and third seconds of its flight.

Ex. 24. A stone is thrown from a cliff with a velocity of 40 ft. per sec. at an angle of depression of 20° (below the horizontal); find the magnitude and direction of its velocity after 2 secs.

41. Newton's Second Law of Motion. We have used the motion of a projectile to discover and illustrate the effect of a constant force on a body in motion in any direction, because it is practically important and experiments on it are easy to make. We can extend our results to include all cases in which a constant force acts on a body, as follows. We usually talk of a body being "at rest" when it lies on the ground—but the ground in England is moving due east in a circle round the axis of the earth at a speed of about 600 miles an hour, while the earth moves in its orbit round the sun. To make it move in this complicated way, a body lying on the ground must be acted on by some force which is continually changing in magnitude and direction. Hence, if the effect of a force on a body depended in any way on the velocity which that body already possessed, or on other forces acting on it, then all our experiments would have given different results every time we repeated them. As they always gave the same results we may assume that each one of a group of forces acting on a body produces its own acceleration independently of the others. So the results which we have found for a force acting alone on a body initially at rest hold good for each one of a group of forces acting on a body, however the body is moving when the forces begin to act;

an acceleration is produced in the body in the direction of the force, and its magnitude is connected with the magnitude of the force and the mass of the body by the equation $F = ma$, as shown in Art. 27.

Newton expressed this fact in his Second Law of Motion, which we may state in the following way. **If a constant force acts on a body, it produces an acceleration in its own direction which is directly proportional to the force and inversely proportional to the mass of the body.** This is a more general statement than the equation $F = ma$, since it is true for all units.

For example, suppose that a mass of 4 lbs. is moving over a smooth horizontal plane (such as a sheet of ice) with a velocity of 10 ft. per sec. in the direction AB, and that a constant

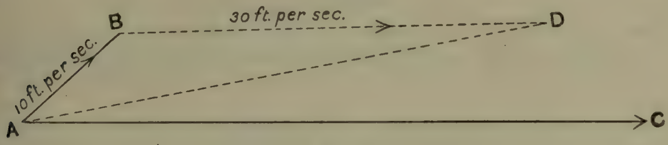


Fig. 30.

horizontal force of 60 poundals acts on it for 2 seconds in the direction AC. 60 poundals produces an acceleration of $\frac{60}{4}$ or 15 ft. per sec. per sec. in a mass of 4 lbs., i.e. in each second it gives it a velocity of 15 ft. per sec., so in 2 secs. it will give it a velocity of 30 ft. per sec. So this is the change of velocity produced by the force, and it will be in the direction AC; the final velocity of the body can be found by compounding the two velocities, 10 ft. per sec. and 30 ft. per sec., which gives AD as the final velocity.

So if two or more forces act simultaneously on a body, the separate changes of velocity produced by the various forces must be compounded together if we wish to determine the final velocity of the body. It follows that we are justified in compounding forces by the triangle or polygon method, in dynamics as in statics, and in calculating the motion of the body as if it were acted on by the resultant force alone. (See Ex. 25.)

Ex. 25. A body of mass 5 lbs. initially at rest is acted on by two horizontal forces, 20 poundals to the E. and 15 poundals to the N. Determine the velocity each force would give it in one second if acting alone; compound these, and find whether this resultant velocity is the same as the velocity produced in one second by the resultant of the two forces.

Ex. 26. A body of mass 2 lbs. rests on a smooth plane inclined at an angle of 30° to the horizontal; it starts with a speed of 10 ft. per sec.

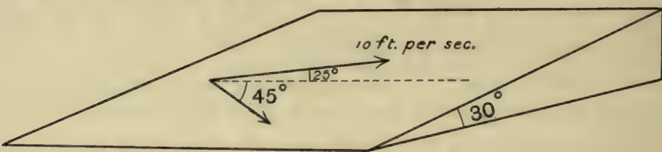


Fig. 31.

obliquely up the plane at an angle of 25° to a horizontal line in the plane, and is acted on by a constant force of 16 poundals acting along the plane at an angle of 45° to the horizontal, as shown in Fig. 31. Find its velocity after $\frac{1}{2}$ sec.

The body will be pulled down the plane by the component of its weight (which is $2 \times 32{\cdot}2$ poundals) parallel to the plane; this component will be found to be (either by calculation or drawing) 32·2 poundals. The force of

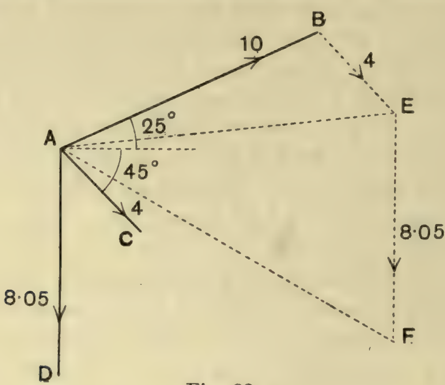


Fig. 32.

16 poundals will produce in $\frac{1}{2}$ sec. a velocity of $\frac{16}{2} \times \frac{1}{2}$ or 4 ft. per sec. in the body, and the force of 32·2 poundals will produce a velocity of $\frac{32 \cdot 2}{2} \times \frac{1}{2}$ or 8·05 ft. per sec.

These may be represented by the lines *AB*, *AC* and *AD* lying in the inclined plane as in Fig. 32; the resultant of *AB* and *AC* (or *AB* and *BE* where *BE* is equal and parallel to *AC*) will be *AE*, and the resultant of *AE* and *AD* will be *AF*, which represents the actual velocity of the body after the two forces have acted on it for half a second. The magnitude and direction of the velocity can be determined from the scale drawing.

Miscellaneous Exercises.

(*g may be taken as* 32 *ft. per sec. per sec., and friction may be neglected.*)

Ex. 1. A body is projected horizontally with a velocity of 30 ft. per sec.; determine its position after ·5, 1, 1·5, and 2 secs. from the start. Draw the path of the body, to a scale of 1 inch to 10 ft.

Ex. 2. A man fires a bullet with a velocity of 2200 ft. per sec. from a rifle held horizontally 5 ft. from the ground; at what distance from his feet will the bullet hit the ground?

Ex. 3. A jet of water moves 5 ft. (measured horizontally) while dropping 1·44 ft.; what was the initial horizontal velocity, and what the final velocity?

Ex. 4. A shot is fired horizontally at 1400 ft. per sec. from a tower 100 ft. high. Find the time of flight and the distance from the foot of the tower at which it strikes the earth.

Ex. 5. A gun is fired at an elevation of 6° with a muzzle velocity of 2000 ft. per sec., the muzzle being 20 ft. above the sea. Find (1) the time which will elapse between firing and the instant when the shell is again 20 ft. above the sea, (2) the time between the instant when the shell is again 20 ft. above the sea and the instant of striking the water; hence find (3) the width of the area in which it would be dangerous for a ship to be whose hull extends 20 ft. above water line.

Ex. 6. An aviator travelling at 60 miles an hour at a height of 800 ft. wishes to drop a bomb on a certain spot; at what horizontal distance from the spot must he release it?

Ex. 7. A stone is thrown at an elevation of 40°, with a speed of 24 ft. per sec. How long will it be in the air, and how high will it rise? What will be its position and speed after half a sec.?

Ex. 8. A man running due E. at 20 ft. per sec. throws a ball with a speed of 30 ft. per sec. so that, if he had been at rest when he threw it, it would have travelled 30° to the W. of S. Find the magnitude and direction of the actual velocity of the ball.

Ex. 9. A man walks, facing towards the W., at 4 miles an hour across the deck of a steamer which is travelling due N. at 10 miles an hour. In what direction and at what pace is he actually travelling with respect to the earth?

Ex. 10. A stone is dropped from the window of a train into a river 50 ft. below; speed of train 30 miles an hour. Find how far the stone will travel horizontally before reaching the water, and how far the train will then have travelled.

Ex. 11. A boat is rowed, so that it would travel at 6 miles an hour in still water, across a stream. To travel straight across the boat has to be headed in a direction inclined at 60° to the bank. Find the pace of the stream. If the stream is 200 yds. wide, how long will it take to cross it? Find the time it takes to cross it if the boat is headed straight for the other shore, and the distance it is carried down stream in the latter case.

Ex. 12. A train is moving N.N.W. with a speed of 30 miles an hour; what is its speed to the N.?

Ex. 13. A bird 50 yds. away is flying N. at 40 miles an hour, and a man due W. of the bird fires straight at it. If the shot travels at 1200 ft. per sec., how far behind the bird will the shot pass?

Ex. 14. The record throw for a cricket ball is 127 yds. 1 ft. 3 ins. Find the least initial velocity that the ball could have received, assuming that the maximum range is given by throwing at an elevation of 45°.

Ex. 15. What record would be set up on the moon, if the initial velocity of the cricket ball were the same as in Ex. 14, given that the acceleration of gravity there is 4·9 ft. per sec. per sec.?

Ex. 16. A bullet is fired from a rifle 4 ft. above the ground with a muzzle velocity of 2000 ft. per sec., so as to strike a target at 500 yds. at the same height of 4 ft. above the ground. Show that the rifle must have an elevation of about 20 minutes, and that a man 6 ft. 2 ins. in height could stand safely half way between the target and the rifle.

Ex. 17. A shot is fired from a gun at an elevation of 20°, with a muzzle velocity of 2200 ft. per sec. What are the horizontal and vertical components of this muzzle velocity? How long will it take the shot to reach the top of its flight?

Ex. 18. A gun is fired at an elevation of 30° and the range is found to be 11,500 yds. Find the muzzle velocity in ft. per sec.

Ex. 19. A sledge party is travelling due N. by the stars at the rate of 10 miles a day over an ice-floe which is drifting S.W. at 15 miles a day. Find the direction and speed at which the party is actually moving.

Ex. 20. A boat is rowed at right angles to the banks of a straight river, at a speed through the water half as fast again as the stream flows. It reaches the opposite bank 2 miles below the starting point. Find the breadth of the river.

Ex. 21. A rider in a circus is standing on a horse which is going steadily at 12 miles an hour, and he has to jump through a hoop whose centre is 4 ft. above the horse's back, and to land on the horse again as before. In what direction should he jump, what should be his initial vertical velocity, how long will he be in the air, how far from the hoop must he be when he jumps?

Ex. 22. A man running straight down the field at 20 ft. per sec. passes a football with such strength that if he had been at rest when he threw it, it would have gone at 30 ft. per sec., and in such a direction that it would have been at an angle of 45° behind the line straight across the field. Determine the direction in which the ball actually moves; is it a "forward" pass?

Ex. 23. Water issues from a nozzle, whose diameter is $\frac{1}{8}$ inch, at an elevation of 60°, and travels a horizontal distance of 15 ft. before it returns to the same level. Given that 1 cub. ft. of water weighs $62\frac{1}{2}$ lbs., calculate the weight of water issuing per minute.

Ex. 24. An arrow shot from a bow has a range of 300 yds., and just clears a tree 60 ft. high when at its greatest height. Find its velocity at starting.

CHAPTER V

WORK AND POWER

42. Useful Work. When a force acts on a machine or a body, it may or may not do useful work. For example, suppose the work to be done is to raise the anchor of a yacht; if you exert a force on the capstan bar, but are unable to exert sufficient force to move the anchor, you do not do any useful work. Again, suppose steam is admitted into the cylinder of an engine which has stopped "on a dead centre," the steam will exert a pressure on the piston, but however large this pressure may be, it will not do useful work until the piston is in a position to yield to the pressure.

In the former case, you are doing useful work when you are exerting a force which successfully overcomes the weight of the anchor, that is while you exert a force on a body which yields to the force; in the latter case, the steam does useful work while it successfully overcomes the resistances which oppose the motion of the engine.

Contrast with this the force which the table exerts on a book lying on the table; it supports its weight, and to that extent it is "useful," but it cannot be said to be "doing work," for it is only exerting a passive resistance. In the same way, when a train is running along a level line, although the rails exert an upward pressure on it to support its weight, they cannot be said to be doing useful *work*; the train is moving, but not in such a direction as to yield to the pressure of the rails, which is again a passive resistance.

So we see that a force does not do work unless its point of application moves, and the movement must be in the direction of the force; we can now give a precise definition of the term.

Work is done by a force when its point of application moves in the direction of that force.

We can decide on any method of measuring it; but the most simple happens to be the most suitable method, and we measure the work done by the product of the force and the distance moved in the direction of the force. The unit of work naturally depends on whether we are expressing force and distance in British or C.G.S. units; if the former, the unit of work is called a "foot-poundal," and is the work done by 1 poundal when its point of application moves 1 foot; if the latter, the unit of work is called an "erg," and is the work done by 1 dyne when its point of application moves 1 cm. Engineers commonly take as the unit the work done by a force of 1 lb. wt. when its point of application moves 1 foot; this is called a "foot-pound."

Ex. 1. A weight of 56 lbs. is pulled up by a rope through a vertical height of 20 ft.; how much work is done on it?

The force exerted by the rope is $56 \times 32 \cdot 2$ poundals; so the work done is $56 \times 32 \cdot 2 \times 20$ ft.-poundals. Expressed in engineer's units, the work is 56×20 ft.-lbs.

Ex. 2. A weight of 5 kilogrammes is pulled up steadily through a vertical height of 3 metres; how much work is done on it?

Ex. 3. A ton of coal is lifted from the bottom of a mine 500 ft. deep; how much work is done?

In these examples we assume that the upward pull of the rope is equal to the weight of the body; so there will be no resultant force on the body and it must be moving with uniform speed. In order to get up this speed at the beginning of the motion we know that the pull of the rope must have been greater than the weight of the body (if m lbs. be the mass of the body and a ft. per sec. per sec. be its acceleration, the pull must have been greater than the weight by ma poundals). But in this case and until further notice we consider the work done during the time while the speed is constant; later we will deal with the work done while the body is getting up or losing speed. We shall assume then for the present that the body is moving steadily; whether its speed is great or small makes no difference to the work done during a definite displacement, though of course the work is done in a shorter time if the speed is high.

43. Connection between the ft.-lb. and the ft.-poundal. A body whose mass is 10 lbs. is attracted to the earth by a force which would give it an acceleration of 32·2, or g, ft. per sec. per sec. if this were the only force acting on it; so this force is $10 \times 32{\cdot}2$ or $10g$ poundals. If it is lifted steadily through a vertical height of 2 ft., the work done by the lifting force is $20g$ ft.-poundals. Since the weight of the body is 10 lbs. wt., the work done on it is 20 ft.-lbs. Hence we see that the same work is expressed by $20g$ ft.-poundals and 20 ft.-lbs.; and, generally, if the value of a certain amount of work is expressed in ft.-lbs., its value in ft.-poundals is found by multiplying by g. Similarly we can express the value in ft.-lbs. of a certain amount of work expressed in ft.-poundals by dividing the latter number by g, or 32·2.

It is convenient to express a result in both units, if the British system is used.

Ex. 4. A train weighs 180 tons and runs along a level line at 30 miles an hour; the resistance to its motion caused by axle friction, air resistance, etc. may be taken as being 10 lbs. wt. a ton. Find the work done by the engine in half a minute.

The force is 180×10 or 1800 lbs. wt.; in half a minute it moves through $\frac{1}{4}$ mile or 1320 feet; hence the work is 1800×1320 ft.-lbs., or $7{\cdot}65 \times 10^7$ ft.-poundals.

44. Work done by a force oblique to the direction of motion. Suppose that you are dragging a stone along a rough surface by means of a rope as in Fig. 33. The movement of the stone is not in the direction of the force, or rather it is not altogether in that direction; so we must extend our definition of work and its measurement to cover such a case. We can do so in two ways, which give the same result; either by saying "the work is measured by the product of the force and the component of the displacement in the direction of the force," or "the work is measured by the product of the displacement and the component of the force in the direction of the displacement." It is assumed, as usual, that the resolution is to be made into two components at right angles to one another.

For example, suppose the force is 1500 poundals inclined at 30° to the horizontal, and that the stone is dragged along the ground horizontally for a distance of 20 feet. If we resolve the force into a horizontal and a vertical component, either by scale

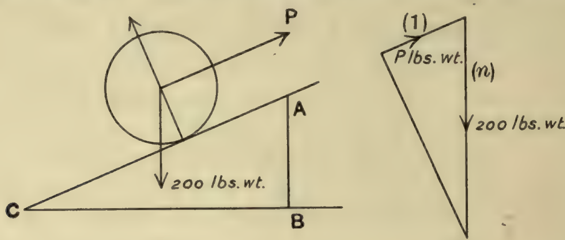
Fig. 33.

drawing or by trigonometry, the former component is found to be 1500 cos 30°, about 1299 poundals; hence the work done is 1299 × 20 or **25,980** ft.-poundals. If we resolve the 20 ft. displacement along and perpendicular to the direction of the force, the former component will be found to be 20 cos 30°, about 17·32 ft.; hence the work done is 1500 × 17·32 or **25,980** ft.-poundals, as before.

It will be seen that according to this definition no work is done by the component of the force perpendicular to the displacement.

45. Work done in raising a weight by different paths. Take the case of a garden roller resting on a slope, as in Fig. 34. Let the gradient of the slope be 1 in n (i.e. $AB = 1$ ft.

and $AC = n$ ft.), and let the weight of the roller be 200 lbs. Suppose the roller is steadily dragged up the slope by a force P lbs. wt. parallel to the surface of the sloping ground and that the frictional resistances may be neglected. We will first find what this force P must be. Consider the forces that act on the roller; they are (i) its weight 200 lbs. wt. vertically downwards, (ii) P lbs. wt. parallel to the slope (these act through the centre of the roller) and (iii) some force exerted by the ground. This last must also act through the centre of the roller, since all three forces must meet in a point; hence it is at right angles to the slope. Since there is no acceleration, the forces on



(i) (ii)

Fig. 34.

the roller must be in equilibrium; so if we resolve (i) along and perpendicular to the slope, the component along the slope must equal P lbs. wt.; hence $P = \frac{1}{n} \times 200$ lbs. wt. (See Fig. 34 (ii) for the force diagram, which is a triangle similar to ABC.)

Therefore as the roller moves up the slope, the "tractive force" P does work at the rate of $\frac{1}{n} \times 200$ ft.-lbs. for each foot it moves up the slope, or 200 ft.-lbs. for each n ft. it moves up the slope. But when it moves n ft. up the slope it rises through a vertical height of 1 ft. Hence the work done in raising the roller through 1 ft. vertically is the same whether it is lifted vertically or rolled up a slope of 1 in n.

Next suppose that the force P, instead of being parallel to the surface of the ground, is inclined to it as in Fig. 35. As before, the component of the weight of the roller parallel to the slope is $\frac{1}{n} \times 200$ lbs. wt. Suppose that we resolve P along and perpendicular to the slope; since there is no acceleration, the former component must be equal to $\frac{1}{n} \times 200$ lbs. wt., and this is the only part of the "tractive force" which does work as the roller moves. Hence as before the work done is the same for a given vertical rise whatever be the gradient of the slope. The other component of P, which is perpendicular to the slope, reduces the pressure of the roller on the ground, but by our definition does no work on the roller as it moves along the ground.

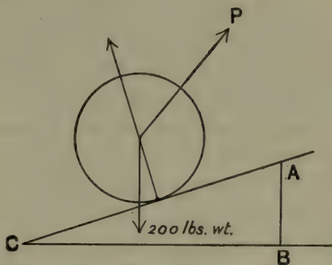


Fig. 35.

The same reasoning can be applied in the case of a body being dragged up a perfectly smooth slope; if however there is a frictional resistance to be overcome, the tractive force is increased and the work done will be greater than in the case of a direct vertical lift.

46. You may be quite ready in a general way to agree that a force can do no work unless it causes the body on which it acts to move to some extent in the direction in which it acts, and yet you may hesitate to accept this definition as reasonable in some particular cases. Suppose for example that a boat (A in Fig. 36) has to be towed up a river by a man B walking on the towing path, and that the tow-rope is of such a length that it makes an angle 30° with the direction of motion of the boat. Suppose the tension of the rope is 15 lbs. wt.; then resolving this force into two components at right angles, that in the direction of motion is 15 cos 30° or 13 lbs. wt., and the other is 15 sin 30° or 7·5 lbs. wt. The man keeps up a pull of 15 lbs. wt., and so might expect to do 15 ft.-lbs. of work for every foot he moves along the tow-path, yet by our definition he only does 13 ft.-lbs. The reason is that the second component of the force he exerts (i.e. 7·5 lbs. wt.) is wasted in a useless effort to pull the boat towards the bank, which is frustrated by the rudder; in ordinary language we say that "he is working at a disadvantage," by which we mean that only part of the force which he exerts does work when he moves, and that the other part of the force does not succeed in effecting anything, i.e. does no "work." If he uses a longer tow-rope he will not work at such a disadvantage;

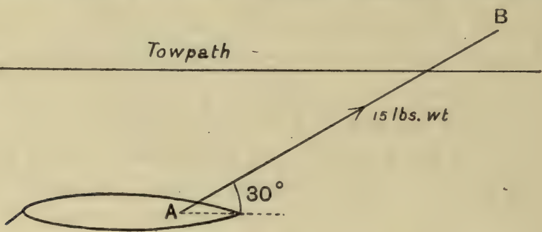


Fig. 36.

he need not exert so large a force, because the component of that force which is used in doing work is now a larger fraction of the force he exerts. But in both cases the actual work which he does is the product of the distance he moves and the component of the force he actually exerts in that direction.

47. We may anticipate a possible difficulty by carrying the enquiry to a further stage. Not only does the component perpendicular to the bank do no work, but it compels the man to do more work than would suffice to pull the boat up stream if the tractive force were applied to the best advantage. For the rudder has to be put on, and this increases the boat's resistance, and consequently the component of the force up the stream necessary to keep it moving. Nevertheless it is still true that the work actually done by the man is the product of this component and the distance he moves.

Contrast with this the example given in Art. 44 of a man dragging a stone over a rough horizontal surface. Here the component of the tractive force perpendicular to the surface reduces the pressure between the stone and the surface, and so decreases the frictional resistance to the stone's motion. Hence in this case this component does no work, and decreases the work which the man must do to move the stone through a given distance. But the work actually done is as before measured by the product of the distance moved into the component of the force actually exerted.

Again, take the case of a man moving a roller over a soft lawn; assume that the man is so tall that the handle slopes upward at a considerable angle with the horizontal. If he pushes the roller, the vertical component of the force he exerts will increase the effective weight of the roller; if he pulls it, he will reduce its effective weight. In the former case the roller will sink deeper and the horizontal component of the force will be greater, so he will do more work for a given displacement. But the vertical component will itself do no work in either case; it can only cause a change in the horizontal component, and thereby indirectly affect the amount of work done.

48. Friction between dry solids. If a heavy block, such as a 14 lb. weight, is dragged at a steady rate along a horizontal surface, such as a table, some force is needed to keep it moving. The "frictional resistance" to sliding motion can be measured by using a spring balance; the results of such experiments will not agree very accurately among themselves, but we shall find that the following laws are approximately true.

I. The frictional resistance does not depend on the area of the surfaces in contact.

This can be tested by using a rectangular block, whose faces are of different sizes, and measuring the friction when it stands on different faces.

II. The frictional resistance does not depend on the speed of sliding.

III. The frictional resistance is proportional to the normal pressure between the sliding surfaces.

This can be tested by putting weights on the block. For an iron block sliding on a sheet of iron it will be found that the frictional resistance to sliding is about one-seventh of the weight of the block. This is usually expressed by the phrase "the coefficient of friction of iron on iron is $\frac{1}{7}$"; in general the ratio of the frictional resistance, when slipping takes place, to the normal pressure between the surfaces is called the "coefficient of friction."

The pressure between the surfaces can of course be produced by any means, not solely by the weight of one body, and the laws hold equally well; for example, the pressure of the rails at a curve against the flanges of the railway carriage wheels, or of a rope coiled round a post. It is important to realise clearly

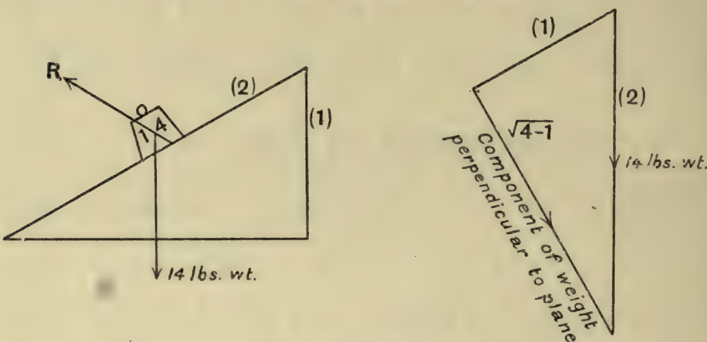


Fig. 37.

that "the pressure between the surfaces" is the component *perpendicular to the surface* of the force with which the two bodies in contact react on one another. For example, suppose an iron 14 lb. weight is slipping down a sloping plate of iron, the gradient of which is 1 in 2, as in Fig. 37. Then the weight is acted on by two forces, 14 lb. wt. vertically downwards, and the force with which the iron plate acts on it. We cannot say that these two forces are equal and opposite, because we do not know that the iron weight is moving with constant velocity; if it has an acceleration, they are not equal and opposite. But whatever this reaction between the bodies may be, it can be resolved into a component perpendicular to the plate and one parallel to it; so also can the weight of the 14 lb. mass. Then we know that the components of these two forces which are perpendicular to the plate must be equal and opposite, for the iron weight has no acceleration in their direction; if one of them were greater than the other the iron weight would not remain in contact with the surface. Hence the normal pressure between the bodies is equal to the component of the weight of the 14 lb. mass perpendicular to the plate. By the triangle of forces, this component is $14 \times \frac{\sqrt{4-1}}{2}$ or 12·13 lbs. wt.

The frictional resistance is, then, one-seventh of this, or 1·73 lbs. wt. (and not one-seventh of 14 lbs. wt.).

By compounding these two components we can if we wish find the total reaction between the iron weight and plate, but this is of small importance.

It should be noticed that the two components parallel to the sloping plate do not balance one another, so the iron weight is slipping down it with an acceleration; for the component of the weight parallel to the plate is $\frac{1}{2} \times 14$ lbs. wt. down the plane and the frictional resistance is 1·73 lbs. wt. up the plane.

Ex. 5. Find the force needed to haul the weight steadily up the sloping plate described above.

The component of the weight down the plane is as before 7 lbs. wt. and the frictional resistance (1·73 lbs. wt.) now acts in the same sense as this component.

Ex. 6. A 14 lb. wt. is dragged steadily up an iron plate sloped to a gradient of 1 in 10; find the necessary force.

Ex. 7. Find the force needed to drag the weight in Ex. 6 steadily down the slope.

Ex. 8. Calculate the work done in moving the weight, in each of the above examples, sufficiently to cause a vertical displacement of one foot.

49. Starting Friction. Take a heavy block at rest on a horizontal table, and attach a spring balance to it; apply an increasing horizontal pull on the spring balance. It will be seen that when the pull reaches a certain value the block begins to move, and that when it is started a smaller pull will keep it moving steadily. Hence the frictional resistance to slipping is greater just before slipping begins than when slipping has begun. This is a matter of some importance in starting or stopping a railway train or motor car; if the brakes are put on too hard, so that the wheels slip, the maximum retarding force of friction is not used, and if in starting a heavy train the steam is admitted into the cylinders too rapidly, the driving wheels may fly round and the engine will not exert so great a tractive force as if the wheels were just on the point of slipping.

The greatest tractive force which an engine can exert on a train is equal to the maximum force of "sticking" friction between the driving wheels and the rails; this depends on the load on the driving wheels. Hence it is important that the driving wheels should carry as much as possible of the weight of the engine. Again, in order to stop the train rapidly, brakes should be fitted to the wheels of every coach, and not only on the engine.

Ex. 9. The load on the driving wheels of a locomotive is 30 tons, and the coefficient of friction between rails and wheels is one-sixth. What is the greatest force it can exert? If its mass is 50 tons, what is the greatest speed it can attain in 10 secs.?

Ex. 10. If the engine of Ex. 9 is hauling a train of mass 100 tons, and if the engine and train need a force of 10 lbs. wt. per ton to overcome their frictional resistance, what is the greatest speed it can attain in 45 secs.?

The greatest driving force which the engine can exert is $\frac{1}{6} \times 30$, or 5 tons wt. or $5 \times 2240 \times 32{\cdot}2$ poundals. The constant frictional resistance is 150×10 lbs. wt. or $150 \times 10 \times 32{\cdot}2$ poundals. So the resultant accelerating force is $(11200 - 1500) \times 32{\cdot}2$ or $9700 \times 32{\cdot}2$ poundals. The mass to be accelerated is 150 tons or 150×2240 lbs., hence the acceleration is $\frac{9700 \times 32{\cdot}2}{150 \times 2240}$ ft. per sec. per sec., and the speed at the end of 45 secs. is $45 \times \frac{9700 \times 32{\cdot}2}{150 \times 2240}$ or **41·84** ft. per sec.

Ex. 11. If the brakes on an engine and train act on all the wheels, how soon can it be brought to a standstill when travelling at 60 miles an hour? (Coefficient of friction $= \frac{1}{6}$.)

The maximum braking force is one-sixth of the weight, so the acceleration is $-\frac{1}{6} \times 32{\cdot}2$ ft. per sec. per sec. Hence the time needed to destroy a speed of 88 ft. per sec. is $\frac{88}{5{\cdot}37}$ or **16·4** secs. During this time its average speed is 30 miles an hour, so it cannot be pulled up in less than an eighth of a mile, even if the rails are dry.

50. Friction of lubricated bearings. Frictional resistance to the sliding of one body on another is greatly reduced if a film of oil is maintained between them. Under these circumstances, the frictional resistance does not follow the same law as with dry surfaces; it increases with the speed of rubbing and with the area of contact, but does not depend on the pressure. But in practice, where the pressure is large a thicker oil, or even grease, is needed than when the pressure is small, and this increases the resistance; for example, when a ship is launched the ways down which it slides are lubricated with a mixture of soft soap and Russian tallow. The commonest instance of lubrication of surfaces sliding on one another occurs in bearings, and in the majority of cases the lubrication is so imperfect that the laws of friction between dry surfaces may be assumed to be followed, except that the coefficient of friction is much smaller, a usual value being ·02 or ·03 instead of about ·15.

Take as an example of this a dynamo with a shaft whose diameter is 1·5 inches, running in bearings, and suppose that each of the bearings has to support a pressure of 1 cwt. Take the coefficient of friction as ·02. The total frictional resistance to the turning of the shaft in its bearing is then $\cdot 02 \times 112$ or 22·4 lbs. wt., exerted along the circumference of the shaft. If the shaft turns through one revolution a point on the circumference moves $\pi \times 1{\cdot}5$ inches or $\frac{\pi \times 1{\cdot}5}{12}$ ft.; so the work done in each bearing against frictional resistance in one revolution is $22{\cdot}4 \times \frac{\pi \times 1{\cdot}5}{12}$ ft.-lbs.

There may be some difficulty in realising the meaning of "the work done in one revolution of the shaft," since in all the instances we have discussed hitherto the displacement has been along one straight line. But the definition of work covers displacement along any line, straight or curved. For example, if men are turning a capstan by pushing the capstan bars, walking round the capstan as it turns, the work they do is measured by the product of the force which they exert and the distance they have walked. In this case, and in that of the shaft turning in its bearings and overcoming frictional resistance, the displacement is always in the direction of the force, so the work is the product of the two. A man riding a bicycle does not push the pedal exactly in the direction it is moving, i.e. at right angles to the crank, so the work he does is less than the force he exerts multiplied by the distance the pedal moves, but if he uses his ankles properly it is not much less than this.

51. Use of wheels. Suppose we have to haul a coal truck weighing 10 tons along a level railway line for half a mile. If it is mounted on iron runners the force we must exert will be about $\frac{1}{7} \times 10$ tons wt. (taking the coefficient of friction as one-seventh), or 3200 lbs. wt. This force has to be exerted for half a mile, so the work is $3200 \times 880 \times 3$ ft.-lbs., or **8,448,000** ft.-lbs.

Suppose that the lines are greased for their whole length, reducing the coefficient of friction to ·03. Then the force becomes 672 lbs. wt., and the work **1,774,000** ft.-lbs. This then would mean less work, but is of course not a method which it would pay to adopt.

Suppose that the truck is mounted on wheels whose diameter is 3 ft., the axles of which are 5 inches in diameter, and that these axles run in lubricated bearings, their coefficient of friction being ·03. We can calculate the work done as in the last article, as follows. The circumference of each wheel is 3π ft., so in half a mile it will turn round $\frac{880 \times 3}{3\pi}$ times.

The total pressure on the bearings is 10 tons, so the frictional resistance to the turning of the shafts is $\cdot 03 \times 10 \times 2240$ or 672 lbs. wt., exerted along the circumference of the axles. When the axles turn through one revolution a point on the circumference moves $\pi \times 5$ inches or $\frac{5\pi}{12}$ ft. Hence the work done in one revolution is $672 \times \frac{5\pi}{12}$ ft.-lbs. But in half a mile the wheels and axles turn $\frac{880}{\pi}$ times; so in that distance the work done is $\frac{880}{\pi} \times 672 \times \frac{5\pi}{12}$ ft.-lbs., or **246,400** ft.-lbs.

Hence the work done per half mile in hauling the truck on wheels is less than one-sixth of what it would be with sliders on greased rails, and less than one-thirtieth of what it would be on dry rails. The larger the wheels and the smaller the axles, the less will be the work.

Since the work done in hauling the truck for 880 yds., or 2640 ft. along the line is 246,400 ft.-lbs., the force exerted by the engine which hauls it must be $\frac{246,400}{2640}$ lbs. wt., or **93·33** lbs. wt. (This force is of course much less than the force of friction between the axle and its bearings, which we saw was 672 lbs. wt.) The usual way of expressing frictional resistance to the motion of trains is to say it is "so many lbs. wt. a ton"; as the truck weighs 10 tons, in this case it is $9\frac{1}{3}$ lbs. wt. a ton. When the speed is high, the resistance of the air becomes considerable, and the frictional resistance may be 16 lbs. wt. a ton.

For a bicycle or motor car running at 20 miles an hour on an asphalt road the resistance is about 40 lbs. wt. per ton, and on an ordinary road about 80 lbs. wt. per ton; if the surface is soft the road resistance rises to 100 lbs. wt. per ton. But here the frictional resistance is chiefly in the wind, the tyres and road surface, and hardly at all in the bearings if these are ball bearings. So a 12 stone man on a 28 lb. bicycle would have to overcome a frictional resistance of about $\frac{168+28}{2240} \times 80$ lbs. wt. or 7 lbs. wt. if he went at 20 miles an hour; at a moderate speed the resistance would be only about 4 lbs. wt.

Ex. 12. The load on each of the back wheels of a brougham is 4 cwt.; the diameter of the axle is $1\frac{1}{2}$ inches, and the coefficient of friction is ·03. Determine the work done against friction in one revolution of the wheel.

Ex. 13. If the diameter of the above wheel is 3 ft., find the work spent on friction in its bearing in 1 mile.

Ex. 14. From the result of the last example, find the tractive force on the back axle when the brougham is on smooth level ground.

52. Rate of doing work. Power and Horse Power.

In practice it is important to consider the rate at which work is done. By the use of a suitable machine a comparatively feeble person can exert a very large force, and by taking a long time he can do a great deal of useful work; but if a boat has to be rowed up stream one man may be unable to work fast enough to overcome the current (and it would be better for him to cast anchor) while two men may be able to do so.

The rate at which work is done is called **Power.** We can state the power of an engine as the number of foot-poundals it can do in a second, or ergs in a second, or foot-pounds in a minute, etc.

Ex. 15. A horse pulling a cart exerts a horizontal pull of 110 lbs. wt. and walks at a steady 3 miles an hour. How much work does it do in a minute?

3 miles an hour is $\frac{3 \times 1760 \times 3}{60}$ feet in 1 min.; so the distance he moves in 1 min. is 264 ft. Hence the work done is 110×264 or 29,040 ft.-lbs. The Power of the horse is **29,040 ft.-lbs. per min.**

The power of an engine is usually expressed by comparing it with that of a horse; to prevent disappointment to the buyer of the engine an unusually powerful horse was originally taken as the standard, one which could work steadily at a greater rate than the one quoted in the above example, and could do 33,000 ft.-lbs. per min. So a "one horse power engine" is one which can do 33,000 ft.-lbs. per min.; a "50,000 H. P. engine" is one which can do 50,000 × 33,000 ft.-lbs. per min.; and so on. An average man can do useful work for a fairly long time at the rate of $\frac{1}{6}$ H. P.

Ex. 16. A 50 H. P. engine is pumping water from the bottom of a mine 800 ft. deep; if it is working for 8 hours, and 30 H. P. is actually spent in raising water, how much will it lift?

30 H. P. for 1 min. gives a total of 30 × 33,000 ft.-lbs.; if this is continued for 8 hours (or 8 × 60 mins.) we get a total of 30 × 33,000 × 8 × 60 or 475,200,000 ft.-lbs. To raise 1 lb. of water needs an expenditure of 1 × 800 ft.-lbs. Hence the number of pounds of water raised is $\frac{475,200,000}{800}$ or 594,000 lbs. Since 1 gallon of water weighs 10 lbs., **59,400 gals.** of water will be pumped up.

Ex. 17. A train weighs 200 tons, and is running along a level line at 30 miles an hour. The frictional resistance to be overcome by the engine is 10 lbs. wt. per ton. Find the H. P. at which the engine is working.

30 miles an hour is $\frac{30}{60}$ × 88 ft. per sec. or 2640 ft. per min. The engine is exerting a force of 200 × 10 lbs. wt.; therefore in 1 minute it does 200 × 10 × 2640 or 5,280,000 ft.-lbs. of work. Hence the necessary horse-power is $\frac{5,280,000}{33,000}$ or **160** H. P.

Ex. 18. The train in Ex. 17 has to be hauled at the same speed up an incline of 1 in 100; what will now be the H. P. of the engine?

In addition to overcoming the frictional resistances to motion, the engine has now to lift 200 tons, or 200 × 2240 lbs., through a certain vertical height; in 1 min. it moves 2460 ft. along the rails, so it rises $\frac{1}{100}$ × 2640 ft. or 26·4 ft. The work done in 1 min. on this account is therefore 200 × 2240 × 26·4 ft.-lbs.; hence the H. P. needed for this is $\frac{200 \times 2240 \times 26\cdot4}{33,000}$ or **358·3** H. P. So the total output of the engine is 160 + 358·3 or **518·3** H. P. The importance of keeping the line level will be seen from this example.

Ex. 19. A train weighing 180 tons runs along a level line at 50 miles an hour; the frictional resistance to be overcome is 20 lbs. wt. per ton; find the H. P. required.

Ex. 20. Find the H. P. required to haul the train of Ex. 19 at the same speed up an incline of 1 in 180.

Ex. 21. A man riding a bicycle along a level road at 9 miles an hour is exerting one-tenth of a horse power; find the resistance he is overcoming.

9 miles an hour is 792 ft. per min.; in 1 min. he does 3300 ft.-lbs.; hence the force he exerts is $\frac{3300}{792}$ or **4·17 lbs. wt.**

Ex. 22. A 12 stone man climbing a mountain rises 1500 ft. in an hour; find the rate at which he is working.

He lifts 12×14, or 168 lbs. through $\frac{1500}{60}$, or 25 ft. in a minute. So in 1 min. he does 168×25, or 4200 ft.-lbs.; hence his H. P. is $\frac{4200}{33,000}$ or **·127** (about $\frac{1}{8}$) H. P.

Ex. 23. A 12 stone man runs up stairs, and gets up to the top of a 60 ft. building in a minute; find his rate of working.

Ex. 24. A man and bicycle together weigh 180 lbs.; when "coasting" down a hill of 1 in 36, they keep up a steady pace of 10 miles an hour. Find the power required to keep up 10 miles an hour on the level.

The component of their weight down the slope is $\frac{1}{36} \times 180$ lbs. wt., or 5 lbs. wt. Hence the frictional resistance to motion at 10 miles an hour is 5 lbs. wt. 10 miles an hour is 880 ft. per min., so the H. P. required is $\frac{5 \times 880}{33,000}$.

***Ex. 25.** If the bicycle in Ex. 24 is geared to 64 inches, and the length of the crank is 7 inches, find the average pressure which the rider must exert on the pedals.

The meaning of "the bicycle is geared to 64 inches" is that one revolution of the crank makes the bicycle move forward a distance equal to the circumference of a wheel whose diameter is 64 inches, i.e. $\pi \times 64$ inches or 16·75 ft. The work the rider must do in this distance is $5 \times 16·75$ ft.-lbs., since the resistance to the motion of the bicycle is 5 lbs. wt. Let F lbs. wt. denote the average pressure on the pedal at right angles to the crank (in practice of course the pressure varies to some extent). In one revolution of the crank the pedal moves through a distance of $2\pi \times 7$ inches or 3·66 ft.; so the work done in one revolution is $F \times 3·66$ ft.-lbs. Hence

$$5 \times 16·75 = F \times 3·66,$$

and

$$F = \mathbf{22·9} \text{ lbs. wt.}$$

53. Transmission of Power. In factories and houses we are provided with power in various forms; water-power, steam-power, electric-power, etc. This is made available for use by some form of motor, water motor, steam engine, electric motor. These have a shaft which revolves against an opposing force, thus driving any machine, lifting a weight or however else we wish to employ the power. There are several ways in which we can transmit the power from the motor, some of which we will now consider. If we want to imagine what goes on when power is thus transmitted, the simplest of these methods is the *chain drive*, used in bicycles and some motor cars. Here an endless chain passes round pulleys, one on the shaft of the motor and the other on the shaft of the machine to be driven; one part of the chain between these pulleys is under tension (the "tight" side), and the other part, or side, is slack. Teeth on the two pulleys fit into the links of the chain to prevent its slipping. If we know the tension in the tight side, in poundals, and the speed at which the chain is running, in ft. per sec., we can immediately calculate the work in ft.-poundals transmitted in a second (by multiplying these two numbers together), and if we wish can express it in horse power.

Ex. 26. The tension in the chain of a bicycle is 10 lbs. wt., and it runs at 8 ft. per sec.; what is the H. P. transmitted?

The work done by the chain in 1 sec. is 10×8 ft.-lbs.; hence in 1 min. it does 80×60 or 4800 ft.-lbs. Now 1 ft.-lb. per min. is $\frac{1}{33,000}$ H. P., so the chain transmits $\frac{4800}{33,000}$ H. P.

It is more usual to use a leather or canvas belt instead of a chain; teeth on the pulleys cannot now be used to prevent slipping, and the friction between belt and pulley must be used instead. If one side of the belt is quite slack, as was the case with the chain, there will not be sufficient pressure between the surfaces of the belt and pulley to produce the necessary friction, so the belt must be shortened and so tightened up until it does not slip. But as in the case of the chain there must be a greater tension in the "tight" side than in the "slack" side of the belt, or it will not transmit any power; if in Fig. 38 A is the pulley of the motor and B the pulley of the machine it is driving and T_1, T_2 are the tensions, in lb. wts., of the two sides of the belt, the retarding force along the circumference of A which the motor is overcoming is $T_1 - T_2$ lbs. wt. (for T_2 is helping A to turn and T_1 is resisting it). The work done by the motor in a second will as before be the product of this force ($T_1 - T_2$ lbs. wt.) and the distance run by the belt in a second. We can easily

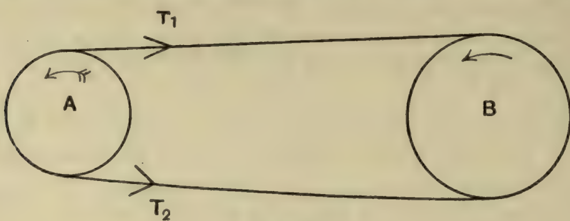


Fig. 38.

determine the latter, by counting the number of revolutions of A in a known time and measuring its circumference, but it is not so simple a matter to determine the difference of tensions in the belt. A method of doing this will be explained later.

In order that the belt should not slip, it may be taken as a rough working rule that the tension in the tight side of the belt must not be greater than $2\frac{1}{2}$ times the tension in the slack side. If then we know the speed of the belt and the power it has to transmit we can calculate the tensions required in the tight and slack sides to avoid slipping.

Ex. 27. The speed of a belt is 2200 ft. per min., and it transmits 50 H.P. Find the tension of each side of the belt.

The work done by the belt in 1 min. is $50 \times 33,000$ ft.-lbs.; in that time the belt travels 2200 ft., so the difference of the tensions must be $50 \times \frac{33,000}{2200}$ lbs. wt., or 750 lbs. wt. Now the larger is $2\frac{1}{2}$ times the smaller tension, so the difference is $1\frac{1}{2}$ times the smaller tension; hence the smaller tension is $\frac{2}{3} \times 750$, or **500** lbs. wt., and the larger tension is **1250** lbs. wt.

Ex. 28. The driving wheel of a motor cycle has a diameter of 28 ins.; the belt pulley on the driving wheel has a diameter of 20 inches; $2\frac{1}{2}$ H. P. has to be transmitted when the bicycle is travelling at 15 miles an hour; find the tensions in the two sides of the belt.

15 miles an hour is 440 yards a minute, or 1320 ft. a minute.

The circumference of the driving wheel is $\pi \times \frac{28}{12}$ ft., so while the bicycle moves 1320 ft. it must turn round $\frac{1320 \times 12}{\pi \times 28}$ times. Hence in a minute the belt pulley turns round $\frac{1320 \times 12}{\pi \times 28}$ times. The circumference of the belt pulley is $\pi \times \frac{20}{12}$ ft., so a point on its circumference (and therefore the belt) moves in 1 min. through $\pi \times \frac{20}{12} \times \frac{1320 \times 12}{\pi \times 28}$ ft., or 943 ft.

Now $2\frac{1}{2}$ H. P. equals $2\frac{1}{2} \times 33{,}000$ ft.-lbs. in 1 min. This must therefore be equal to 943 multiplied by the difference of tension in lbs. wt. So the difference of tension is $\frac{5 \times 33{,}000}{2 \times 943}$ lbs. wt. or 87·5 lbs. wt. As in Ex. 1, this difference is $1\frac{1}{2}$ times the tension in the slack side, so the tensions are **145·8** and **58·3** lbs. wt.

54. Measurement of the Horse Power of a motor by a rope brake. The "H. P. of a motor" is not a definite quantity like its weight; it depends on the work it is set to do, the speed at which it runs, and the amount of explosive gas, electric current, water, etc. which is supplied to it. Suppose that we have to determine the output of a motor when these conditions are known, for example suppose it is a petrol motor driving by means of a belt a dynamo which is supplying current at the proper voltage to a definite number of lamps. If we switch on more lamps, the "load" on the dynamo is increased, the motor has to do more work and will probably slow down unless the "throttle" is opened somewhat so that more petrol passes into it. If, instead of switching on more lamps, we hold the tight side of the belt, so that the belt cannot move and the driving pulley of the motor can only turn by slipping, the motor will slow down as before, since the retarding force on it is increased. Leave the "throttle" unchanged, but gradually reduce the tension in the *slack* side of the belt, thus reducing the pressure between the belt and pulley, and so reducing the friction, until the motor picks up again to the same speed as before. Then, so far as the motor is concerned, it is running under exactly the same conditions as when it was driving the dynamo, so it must be doing work at the same rate. But now the motor is doing work against the retarding force of friction instead of against the resistance of the dynamo; the belt is standing still and we can now easily measure the tensions in each side. These tensions will not be the same as when the motor was driving the dynamo, but since the motor is running at the same pace and doing the same work, the retarding force must be the same as before, i.e. the *difference* of the tensions must be the same as before. (Each must therefore have been reduced by the same amount when the belt was slackened to let the motor pick up its original speed.)

A convenient way of arranging the belt in order to measure this difference of the tensions is shown in Fig. 39. Here the belt is not an endless one; one end carries a pile of weights, the other is attached to a spring balance whose other end is fixed. The weights are adjusted until the motor runs at its original speed with the original supply of petrol (or current, etc., according to the kind of motor it is). Then the difference of the tensions in the tight and slack sides can be found by subtracting the reading of the spring balance from the value of the weights; when the number of revolutions per min. of the motor has been observed by a revolution counter and the circumference of the pulley has been measured, we can calculate the horse power at which the motor was working.

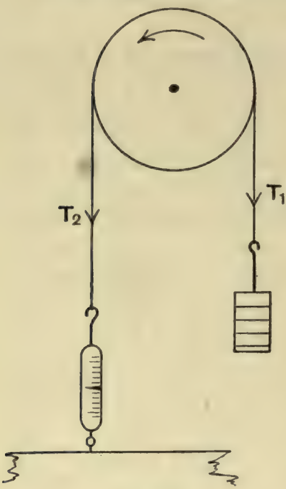


Fig. 39.

This is called a Rope Brake; as it absorbs the power of the motor it is called an Absorption Dynamometer; and the result of the test made with it is called the Brake Horse Power of the engine.

Ex. 29. The motor makes 1200 revs. per min.; the diameter of the pulley is 18 inches; the reading of the spring balance is 19·5 lbs. wt. and the weights on the other side amount to 78 lbs. What is the H. P.?

The circumference of the pulley is $\pi \times 1{\cdot}5$ ft., so in 1 minute the distance which a point on the circumference moves against the resistance is $1200 \times \pi \times 1{\cdot}5$ ft. The frictional resistance is the difference of the tensions, or $78 - 19{\cdot}5$, or 58·5 lbs. wt. Hence the work done in 1 minute is $1200 \times \pi \times 1{\cdot}5 \times 58{\cdot}5$ ft.-lbs., so the H. P. is $\frac{1200 \times \pi \times 1{\cdot}5 \times 58{\cdot}5}{33{,}000}$ H. P. or **10·02** H. P.

Ex. 30. The tensions on the two sides of a rope brake are 8 and 2 lbs. wt.; the diameter of the pulley is 1 ft. and the speed is 100 revs. per min.; calculate the H. P.

55. Work represented by Area. Suppose that a body is acted on by a constant force, and moves in the direction of that force. If we draw a diagram to represent the force at every point of its path, taking abscissae to represent the displacement of the body, and ordinates to represent the force, then this "force-displacement curve" is a straight line (AB in Fig. 40); at any point M the force on the body is represented by PM. Now the work done on the body while it moves from O to M is the product of the force on it and the distance it has moved; the area OP is the product of the length PM and the length OM. Hence we may say that the area OP represents the work done on the body by the force represented by PM when it moves a distance represented by OM. By this we merely mean that the area OP contains as many units of area as the work done contains units

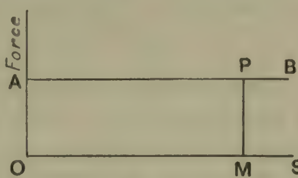


Fig. 40.

of work, and we must be careful to take the correct unit of area. This is decided when we have decided on the scale to represent forces and distances. For example, if we take an inch to represent 1 lb. wt., and an inch to represent 1 ft., then a square inch is our unit of area, and a square inch represents 1 ft.-lb. But if we take 2 ins. to represent 1 lb. wt., and 3 ins. to represent 1 ft., then our "unit of area" in the above statement is 6 sq. ins., representing 1 ft.-lb.; or we may put this latter case in another way, and take 1 inch to represent $\frac{1}{2}$ lb. wt., and 1 inch to represent $\frac{1}{3}$ ft., then 1 sq. in. will represent $\frac{1}{2} \times \frac{1}{3}$ or $\frac{1}{6}$ ft.-lb. This is a more usual way of stating the "scale" of a diagram. If we know this scale, i.e. how many ft.-lbs., ft.-poundals or ergs are represented by 1 sq. in. or 1 sq. cm. on the diagram, we can at once calculate the work done when we have measured the area; and we can calculate what the scale is when we know what scales are used for force and distance, as was done above.

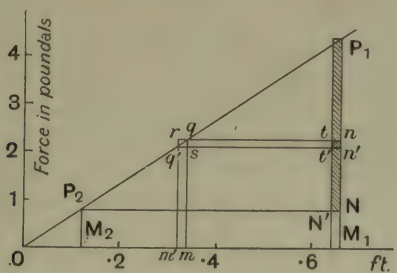
Ex. 31. The scale of a force displacement diagram is 1 in. to 100 lbs. wt., and 1 in. to 20 ft.; what is the area-scale for the work?

Ex. 32. On the above diagram, what work is represented by an area of 2·97 sq. ins.?

Ex. 33. A force of ·7 lb. wt. acts on a body for a distance of 16 ins. Draw the force-displacement curve to some convenient scale of inches to poundals and feet; determine the value of a sq. in. in ft.-poundals, and hence calculate the work done.

56. This introduction of units of area and scales may seem a roundabout business, and as a matter of fact it is a useless complication in dealing with simple areas like that of a rectangle. In those cases it is much more straightforward and simple to find from the diagram, or from the conditions of the problem, the values of the acting force and the displacement and multiply these together to find the work done. But there are many cases, such as we shall consider presently, where we cannot calculate the work in this simple way, but have actually to measure an area on the diagram itself by a planimeter, or in some other practical way, and deduce from our result the value of the work done; in that case we require to know how much work is represented by 1 sq. in. or other unit of area of the diagram.

57. Work done by a variable force. Suppose that a body is acted on by a force which changes in magnitude during the motion, and that we know the value of the force at each point of the body's path (*not* at each instant of the motion). For example, we know that the force exerted by a spiral spring is proportional to the amount it is extended beyond its normal unstretched length; so if a spring has one end fixed and the other attached to a body which moves so as to extend the spring, the force on the body will be proportional to the distance it has moved from the point at which the spring first acted on it. The force-displacement curve is shown as OP in Fig. 41.



1 inch to ·4 ft., 1 inch to 4 poundals, 1 sq. in. to 1·6 ft.-poundals.

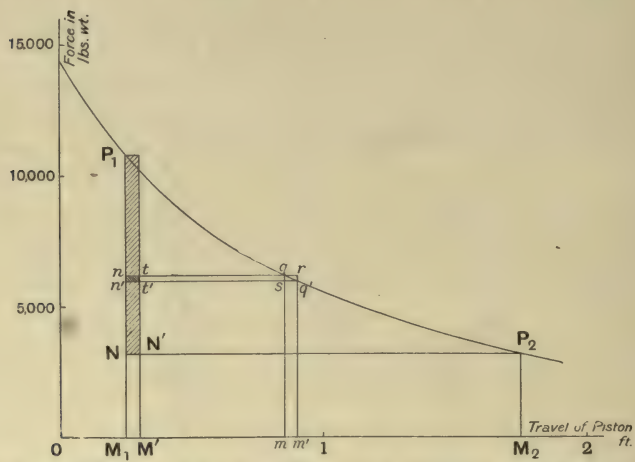
Fig. 41.

Again, if a quantity of gas, at a pressure above that of the atmosphere, is contained in a cylinder fitted with a piston, and the piston is allowed to move outward, doing work as it goes, the pressure of the gas will decrease as the volume it occupies increases. The force exerted by the gas on the piston depends on the position of the piston. Fig. 42 shows the force-displacement curve for steam expanding in the cylinder of a steam-engine, starting with

1 cub. ft. of steam in the cylinder at a pressure of 100 lbs. per sq. in. above that of the air, and assuming that the area of the piston is 1 sq. ft.

In any such case we can compute the work done during any displacement from the area bounded by the ordinates through the points representing the initial and final positions, the curve and the horizontal axis.

For, let M_1, M_2 represent the initial and final positions, and P_1M_1, P_2M_2 the ordinates. Subdivide M_1M_2 into a large number



1 in. to ·74 ft., 1 in. to 7400 lbs. wt., 1 sq. in. to 5476 ft.-lbs.

Fig. 42.

of equal parts, and call mm' one of them, qm and $q'm'$ being the ordinates. Then qm, $q'm'$ represent the forces at the positions represented by m and m'. For example, in Fig. 42 M_1M_2 is divided into 30 equal parts, so that M_1M', mm' represent displacements of $\frac{1}{20}$ ft. each, and the force at m is 6200 lbs. wt. and at m' it is 5960 lbs. wt. Draw the parallel lines as in the figure. If during the step mm' the force had a constant value represented by qm (or 6200 lbs. wt. in this example), the work done during this small displacement would be represented by the area $qrm'm$ (i.e. by $6200 \times \cdot 05$ ft.-lbs.). If during the step the force had remained constant, with a value represented by $q'm'$ (or 5960 lbs. wt. in this example), the work done during the step would be represented by the area $sq'm'm$ (or $5960 \times \cdot 05$ ft.-lbs.). As a matter of fact, the force changes gradually from one value to the other, so the work done must be intermediate between these two values. Hence if we take the area $qq'm'm$ as representing the work done (qq' being the arc of the curve), our error cannot be greater than the work represented by the little rectangle $qrq's$ (or $240 \times \cdot 05$ ft.-lbs.). This rectangle is equal to $ntt'n'$ (shown shaded). Considering all the steps from M_1 to M_2, the work done can be represented by the area bounded by the curve $P_1 P_2$, the ordinates $P_1 M_1$ and $P_2 M_2$, and the straight line $M_1 M_2$, the error in doing so being less than the work represented by the area of the rectangle $P_1 N'$ (shown shaded); for this is the sum of all the little rectangles like $ntt'n'$. (In our example, $P_1 M_1$ represents 10,760 lbs. wt. and $P_2 M_2$ represents 3230 lbs. wt., so the area $P_1 N'$ represents $7530 \times \cdot 05$ ft.-lbs.)

Now the area $P_1 N'$, and therefore the amount of the error, depends on the breadth of the strips mm', etc. which we can make of any size we please; by increasing their number and so reducing their breadth we can reduce the possible error to as small a value as we please. Hence we may assert that to any required degree of accuracy the area $P_1 P_2 M_2 M_1$ represents the work done in the displacement M_1 to M_2.

The value in sq. ins. of the area can be determined in any of the many practical ways available; by counting squares, by Simpson's rule, by measuring 10 ordinates and taking the mean, etc.; then from a knowledge of the scale we can calculate directly the work done during the displacement in ft.-lbs., etc. For example, in Fig. 42 the area $P_1 P_2 M_2 M_1$ is 1·603 sq. ins., so the work done in the displacement from M_1 to M_2 is 8780 ft.-lbs.

Ex. 34. The following are some of the corresponding values of force (in lbs. wt.) and travel (in ft.) of piston from which Fig. 42 was drawn:

Travel	0	·19	·67	1·33	1·92
Force	14400	11520	7200	4320	2880

Plot the curve on a scale of 1 in. to ·25 ft. and 1 in. to 2500 lbs. wt.; determine the area in sq. ins. under the curve for travel of piston from ·5 to 1·5 ft., and deduce the work done in that part of the stroke.

Ex. 35. Determine the area under the force-displacement curve in Fig. 41, for an extension of the spring from 0 to ·5 ft., and deduce the work done, in ft.-poundals.

Ex. 36. Prove that the work done in extending a spiral spring for any distance is the same as if there were a constant force acting through the same distance and equal to half the sum of the initial and final tensions of the spring.

58. Indicator Diagrams. In an ordinary steam engine, the piston starts at the end of the cylinder and steam at high pressure is admitted from the boiler into the cylinder; this exerts a constant pressure on the piston. After the piston has moved for a part of its stroke under the action of this constant pressure, the supply of steam from the boiler is cut off; as the piston moves onward for the remainder of its stroke, the steam in the cylinder expands, exerting on the piston a continually decreasing pressure as described in Article 57. Hence the force-displacement curve will be as shown in Fig. 43, where OP represents the constant pressure in the first part of the stroke (from O to A), and QR is the expansion curve described in Article 57, the pressure decreasing until the piston has reached the point at the end of its stroke represented

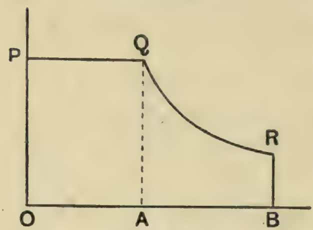


Fig. 43.

by B. The steam is then allowed to flow out into the exhaust, and, of course, does no further work on the piston. Then the work done by the steam during the stroke will be represented by the area $PQRBO$, and it can be calculated when the scale of the diagram is known. Such a diagram is called an Indicator Diagram. It is a matter of every-day practice in testing engines to obtain these diagrams, for not only do they enable us to calculate how much work the steam is doing, but by their difference in shape from the theoretically perfect diagram of Fig. 43 they indicate any faults there may be in the arrangements for admitting and exhausting the steam, etc. By means of a simple piece of apparatus the engine is made to draw its own indicator diagram to a known scale; a piece of paper is made to move with the piston, say from right to left, like the paper attached to the trolley in experiments described above, and a pencil moves up or down the paper, but not to right or left. This pencil is connected to the piston of a little separate cylinder which is connected to the main cylinder and so contains air or steam at the same pressure; the motion of the little piston is opposed by a strong spring, so the height of the pencil above a fixed line OB on the paper shows the pressure of the steam in the piston of the main cylinder. Hence as the paper moves from right to left the pencil traces out the line PQR, remaining at the same height above OB while the steam pressure is constant, but dropping back towards OB as the pressure falls, and tracing the line RB when the pressure of the steam suddenly drops to that of the air, when the steam is exhausted. During the backward stroke the cylinder is still connected to the exhaust, so the pencil traces the line BO as the paper moves from left to right; when steam is again admitted for the next stroke the pencil jumps up to P again, and the cycle is repeated.

59. Average value of a variable quantity. If a quantity varies, there are usually several methods of arriving at its "average" values, each leading to a different value. For instance, in an hour's motor drive in a hilly country, the speed might be observed at the end of each minute, or each mile, and the arithmetic mean of either of these sets of values may be taken as the average; or again, the total distance divided by the total time may be taken. The three methods will in general give different results; the last is the one commonly adopted.

Again, if the force on a body varies, and its value is known to depend on the *position* of the body, we take as the definition of the average force, "*that constant force which, acting on the body during the displacement, does the same work on it.*" For instance, in the case given in Art. 57, where the force does 8780 ft.-lbs. in a displacement of 1·5 ft., the average force is 5851 lbs. wt.; for that constant force, acting through 1·5 ft., does 8780 ft.-lbs. of work. It should be noted that this average force is not the arithmetic mean of the forces at the beginning and end of the displacement, for these are 10,760 and 3230 lbs. wt.

If, however, as in the case of the spiral spring, the force is directly proportional to the displacement, then the average force is the arithmetic mean of the initial and final forces (see Ex. 36, above).

In order to find the average force in other cases, we must draw the force-displacement curve, determine the area under it for the given displacement (i.e. the area bounded by the curve, the ordinates through the points representing the initial and final positions, and the axis of displacement) and draw the equivalent rectangle on the same base; then the height of this rectangle represents accurately the "average force."

A good approximation to this "average force" is got by dividing the displacement into 10 equal parts, drawing the ordinates to the force-displacement curve at the mid-points of these parts, and taking their arithmetic mean as representing the average force. This is one of the practical methods of finding roughly the area under a curve, and is commonly adopted in the case of indicator diagrams. The value found in this way from an indicator diagram is usually called the "mean effective pressure" of the steam, and is the constant pressure on the piston throughout the stroke which would do the same amount of work as is actually done by the varying pressures.

60. Indicated Horse Power. When the mean effective pressure of the steam in the cylinder of a steam engine has been determined from an indicator diagram as described in the last Article, it is easy to deduce the rate at which the steam is doing work. Suppose that this mean effective pressure is P lbs. wt. per sq. in., the length of a stroke is L ft., the area of the piston is A sq. in., and the number of working strokes per min. is N, then the average force on the piston is $P \times A$ lbs. wt., the work done in a stroke is PLA ft.-lbs., the work done in a minute is $PLAN$ ft.-lbs., and the horse power is therefore $\frac{PLAN}{33,000}$. Owing to friction, etc., this is more than the horse power available for practical use, and it is called "indicated horse power" or I. H. P. to distinguish it from "brake horse power" or B. H. P. (see Art. 54).

Ex. 37. A single cylinder single-acting steam engine runs at 150 revs. per min. and so makes 150 working strokes a minute. The piston is 12 ins. in diameter, and the stroke is 15 ins.; the mean effective pressure of the steam is 50 lbs. per sq. in. What is the I. H. P.?

Miscellaneous Exercises.

Ex. 1. A man raises $2\frac{1}{2}$ tons through 9 ins. in 2 minutes by applying a constant force of 44·4 lbs. wt. at right angles to the "tommy-bar" of a screw-jack at a point $2\frac{1}{2}$ ft. from the centre of the jack. The pitch of the screw is $\frac{1}{2}$ inch. Calculate the number of turns of the jack required, hence the displacement of the point of application of the force, and hence the power (in H. P.) at which the man is working. Calculate also the rate at which work is being done against gravity.

Ex. 2. The tensions in the two sides of a belt driving a pulley are 180 lbs. wt. and 100 lbs. wt., and the pulley, whose diameter is 2 ft., runs at 120 revs. per min. Find the speed of the belt and the H. P. transmitted.

Ex. 3. A flat leather belt transmits power from one pulley to another of equal size; we want to have a difference of 90 lbs. wt. in the tensions of the two parts of the belt. What tension will be necessary in the slack part?

Ex. 4. An engine delivers 100 H. P. through a belt on its fly-wheel (diameter 6 ft.) running at 80 revs. per min. Calculate the speed of the belt; and assuming that to prevent slip the tension in the driving side of the belt must not be more than twice that in the slack side, find what the latter must be.

Ex. 5. A 12 stone man carries a load of 32 lbs. up a hill 250 ft. high in 5 minutes; at what horse-power does he work?

Ex. 6. A motor car weighing 2 tons runs up a hill of 1 in 10 at a speed of 20 miles an hour; the resistance due to wind etc. is 60 lbs. wt. What is the useful horse power?

Ex. 7. A four cylinder motor makes 1200 revs. per min. (or 2400 effective strokes). The bore of the cylinders is 4 ins. and the stroke is 6 ins. The mean pressure of the gas is 60 lbs. per sq. in. Calculate the I. H. P.

Ex. 8. A pump delivers 500 gallons of water per minute to a height of 100 ft.; what horse power is it developing? (1 gall. of water weighs 10 lbs.)

Ex. 9. The lengths in inches of equidistant ordinates of an indicator diagram are, respectively, 1·6, 1·94, 1·94, 1·8, 1·4, 1·06, ·82, ·64, ·56, ·4. The stroke is represented by a length of 3·04 ins. on the diagram. Find the area of the diagram in sq. ins. If the scale of the diagram is 1 in. to 50 lbs. per sq. in., and 1 in. to 4 ins. stroke, calculate the mean pressure. If the area of the piston is 30 sq. ins. calculate the work done in a stroke. If there are 100 working strokes per min., calculate the I. H. P.

Ex. 10. A force of 7 lbs. wt. is found to extend a spring 3 ins. Calculate the tension of the spring when extended 5 ins.; hence calculate the work done in extending it 5 ins.

Ex. 11. The I. H. P. of an engine is 115, and its B. H. P. is 100. How much work is spent in a minute in overcoming internal friction?

Ex. 12. If the frictional resistance to the motion of a bicycle and man, which weigh together 180 lbs., when running at 6 miles an hour is 3 lbs. wt., find the power required to propel it at that speed up a hill of 1 in 27.

Ex. 13. A trolley plane, 6 ft. long, is tilted so that one end is 3 inches higher than the other; a trolley (whose mass is 6 lbs.) is then found to run down with uniform speed when started. What is the force of friction acting against the trolley?

Ex. 14. A "hill-climbing formula" for motor cars gives the H. P. as the weight of the car in lbs. multiplied by the vertical rise in ft., added to the length of the hill in ft. multiplied by 40 lbs. wt. per ton of car, and the result divided by 33,000 multiplied by time of ascent in mins. Show that this formula leads to a correct result if the frictional resistance to the car's motion is 40 lbs. wt. per ton.

Ex. 15. A sledge of mass 300 lbs. is being dragged by a horse with uniform velocity along a horizontal road. The two traces make an angle of 20° with the ground and the tension in each trace is 50 lbs. wt. Find (1) the total pressure between the sledge and ground, (2) the force of friction between the sledge and ground, (3) the coefficient of friction. If the horse moves at 3 miles an hour, find the H. P. at which he works.

Ex. 16. 10 H. P. is to be transmitted from one shaft to another by means of a belt. The pulley on the former shaft is 2 ft. in diameter and turns at 100 revs. per min. Calculate (1) the speed of the belt, (2) the difference of tension between the tight and slack sides of the belt.

Ex. 17. The pressure between a shaft and its bearings is 2000 lbs. wt., the diameter of the shaft is 4 ins., and the speed 250 revs. per min. If the coefficient of friction is ·02, find the loss of power in the bearings.

Ex. 18. A locomotive of mass 60 tons is hauling a train of mass 100 tons at a uniform speed up an incline of 1 in 120. Assuming that the frictional resistance to the motion of locomotive and train is in lbs. wt. per ton $6+\frac{V^2}{200}$ where V is speed in miles per hour, find the H. P. at which the locomotive works when the speed is (*a*) 30, (*b*) 40 miles per hour.

Ex. 19. A motor weighing 2 tons is maintaining a steady speed of 30 miles an hour along a level road. On reaching a descent of 1 in 20 the supply of gas is cut off, but the speed still remains 30 miles an hour. Calculate the effective H. P. exerted by the engine when running at this speed on the level.

Ex. 20. A car weighing 1 ton can ascend a certain hill, which rises 150 ft. in $\frac{1}{4}$ mile, at 20 miles an hour. Taking the frictional resistance at 40 lbs. wt., find the H. P. at which the engine is working.

CHAPTER VI

ENERGY

61. Energy. When a body is capable of doing work, it is said to possess *Energy.* Energy is the capacity to do work; it may exist in a body in various forms. For instance, energy is stored in coal; when a steamship or locomotive takes in coal it becomes capable of overcoming the resistance to its motion, and when all its coal is burnt it has no energy left and can do no more work. Blasting powder possesses a store of energy; when it explodes it can do work in breaking coal away from the coal face, which otherwise the miner would have to do with his pick. A heavy weight which is raised up so that it has room to fall possesses energy; in its descent it can lift another weight or drag it along against a frictional resistance. The mainspring of a watch possesses energy when it is wound up (i.e. coiled up tightly), for it can keep the watch going against the frictional resistances of the train of wheels.

The natural method of measuring energy is by the work it can do; hence we express its value in ft.-lbs., foot-poundals or ergs. In dealing with energy in some of its forms, it is sometimes convenient to express its value in other units; for instance, electrical energy is sold at so much a "kilowatt-hour"; but in every such case we can calculate what this represents when reduced to ft.-lbs. of work, though it is not usual to perform the calculation. As we are chiefly interested in mechanical energy, we will express it in ordinary units of work.

62. Conservation of Energy. Energy can be transferred from one body to another, and its form can be changed in various ways. For instance, the energy of coal can be extracted in the furnace of a boiler and transferred to the steam; the steam can be passed on to a steam engine working a pump, which pumps water up to an elevated reservoir. The energy of the steam is thus transferred to the water in the reservoir and is stored there. If this water is made to drive a turbine or water wheel at a lower level, and the turbine drives a dynamo, the energy passes with the electric current to the lamps, and appears there as heat energy, or if the lamps are not in use at the moment the current is diverted into a battery of accumulators and the energy is stored there until it is needed, or it may be used in driving an electric tram. Throughout all these transfers from one body to another and transformations from one form to another, *there is no change in the amount of the energy.* This has been proved by a series of careful experiments extending over many years, and it is now accepted as a Law of Nature; it is of the utmost value in practical life. It is usually called the **Law of Conservation of Energy.**

It is necessary here to guard against misunderstanding the above statement. When the energy of the coal was devoted to driving the steam engine, some but not all of the heat energy was transformed into mechanical energy; some of the heat passes up the chimney, for example. No heat engine will convert all the heat energy supplied to it into mechanical energy, but that part which is so transformed is exactly equal in amount to the heat energy which has disappeared in the process. In fact if the mechanical energy so produced is turned back into heat energy (and this change can easily be made complete) we regain exactly the amount of heat energy which disappeared. Hence in the transformations of energy which we have described we must not expect that all the energy of the coal will reappear in the electric lights; some remains over at each step, and what reaches the lamps is only a fraction of what was used at the start. The meaning of the law of Conservation of Energy is that if all these remnants are taken into the account, the sum total of energy remains unchanged in the changes which the energy has undergone.

It must also be understood that the bodies which serve as carriers of the energy during its changes are not interfered with from outside the group or "system"; they must not do work on other bodies or have work done on them by other bodies. We can, of course, secure this condition by making the system large enough to include all such bodies as can interfere with those we are actually experimenting on. The law can be stated in the following way. "*The total energy of any system is a quantity which can neither be increased nor diminished by any action between the parts of the system, though it may be transformed into any of the forms of which energy is susceptible.*"

63. Joule's Equivalent. It would be out of place in a book on Dynamics to deal at length with the various forms of energy and the way it is measured. But as work done against friction always results in the production of heat energy, we will take this transformation as an example of the application of the law. The first experiments to establish the law were made on the change of mechanical into heat energy by Dr Joule of Manchester, on somewhat the same lines.

If for the pulley over which passes the rope brake described in Art. 54 we substitute a cylindrical drum of thin brass, containing cold water, it will be found that as the drum turns the water gets steadily warmer. The rise of temperature is observed by a thermometer, and we know that the heat produced is measured by the product of the mass of the water and the rise of temperature. Allowance has to be made for the heat used in warming the metal of the drum; but this can be done by adding to the mass of the water what is called the "water equivalent" of this mass of metal, which is determined by a separate experiment.

And in dealing with heat it is difficult to secure the non-interference of surrounding bodies, which we said above it is essential to secure; for example, we have to be sure that no heat energy passes by radiation between the drum and the walls of the room. As a matter of fact we cannot prevent its doing so, but we can either calculate the amount which does so, and allow for it, or what is simpler we can safely neglect it by arranging matters so that the final temperature is as much above that of surrounding bodies at the end of the experiment as it was below it at the beginning. In that case the gain of heat from surrounding bodies practically balances its loss to them.

Under these conditions it is found that the generation of heat proceeds at a uniform rate if the drum is turned at a uniform rate, and that in experiments with different rates of turning, different quantities of water, etc., for every foot pound of work done there is the same quantity of heat produced, thus verifying the law in this instance.

Ex. 1. In an experiment of this kind the tensions on the two sides of the belt were 5·6 and ·5 lbs. wt.; the diameter of the drum was 6 ins.; temperature of the water at beginning was 56° F. and after 680 revolutions was 63° F.; the weight of water, including the water equivalent of the drum, was ·98 lb. Calculate the work done, and the heat produced.

The difference of the tensions was 5·1 lbs. wt. The circumference of the drum was $\pi \times 6$ ins., so a point on its circumference moved $\pi \times 6 \times 680$ ins. or 1068 ft. Hence the work done was $5·1 \times 1068$ or 5447 ft.-lbs.

The rise of temperature was $63 - 56$ or 7° F. So the heat produced was $·98 \times 7$ or 6·86 British Thermal Units (one B. TH. U. is the heat which will raise 1 lb. of water through 1° F.). Hence 1 British Thermal Unit is equivalent to $\frac{5447}{6·86}$ or 794 ft.-lbs.

Accurate experiments of this kind show that one British Thermal Unit is evolved whenever **778** ft.-lbs. of mechanical work are converted into heat energy. This number is usually called Joule's Equivalent. By its means we can immediately transform into ft.-lbs. the value of heat energy stated in British Thermal Units.

64. Potential Energy. A weight raised above the ground can do work in descending to a lower level; it is an example of a body possessing a store of mechanical energy. When a body possesses mechanical energy in consequence of its position, or its configuration (as in the case of a stretched or bent spring), it is said to have *Potential Energy*.

As an example of the way in which this potential energy is measured, take the case of a weight of 10 lbs. hanging on the chain of a set of pulleys, or a Weston block, or some other machine which is arranged to lift a very heavy weight or overcome a resistance in some other way. If this weight is just sufficient to overcome the resistance of the machine (including its frictional resistance) it will run down at a steady pace when started, exerting a force of $10 \times 32{\cdot}2$ poundals. If it descends 4 ft. before reaching the floor, it will do $10 \times 32{\cdot}2 \times 4$ or 1288 ft.-poundals of work. So this is the work it was capable of doing when at its starting level; hence its potential energy was then $10 \times 32{\cdot}2 \times 4$ ft.-poundals.

In general, if M lbs. be the mass of the body, S ft. the height through which it can descend, then its potential energy is MgS ft.-poundals.

Consider what becomes of the energy originally in the raised weight of 10 lbs. which we have taken as an example. Work to the extent of 1288 ft.-poundals is done on the machine. Let us assume that the velocity ratio of the machine is 4, and that the load which an effort of 10 lbs. wt. can raise is 25 lbs. wt. Then when the 10 lbs. weight drops 4 ft., the 25 lbs. weight rises 1 ft.; hence $25 \times 32{\cdot}2 \times 1$ ft.-poundals of work are done on this weight, and it acquires potential energy amounting to $25 \times 32{\cdot}2 \times 1$ ft.-poundals, or 805 ft.-poundals. The remainder of the 1288 ft.-poundals of work done by the descending weight is spent in overcoming the friction of the machine; this produces an equal amount of heat energy. Hence the potential energy of the 10 lb. weight is transformed partly into potential energy in another weight, partly into heat energy.

Ex. 2. A waterfall is 500 ft. high. Each pound of water falls 500 ft., and loses 500 ft.-lbs., which are converted into heat, yielding $\frac{500}{778}$ B. TH. U. If all the heat is retained in the water, the water will be $\frac{500}{778}$° F. warmer at the bottom of the fall than at the top.

65. Kinetic Energy. Suppose that the 10 lb. weight, described in the last Article, is allowed to fall freely, not driving any machine and with nothing to check its fall. It cannot transfer its energy to any other body, nor change it into heat as it falls, and yet as it loses height it loses its potential energy. By the law of conservation of energy we know that its energy cannot disappear; hence it must remain in the body in some form other than potential energy. As the body falls and loses its potential energy, it gains speed; hence it would seem that a body in motion must possess energy merely because it is in motion.

The energy which a body possesses in consequence of its motion is called *Kinetic Energy*.

In the case we have been considering, of a body falling freely from rest, we can calculate how much kinetic energy it possesses, when we know its mass and the distance it has fallen, because we can calculate how much potential energy it has lost; by the law of conservation of energy this must be the amount of kinetic energy it has gained. In the case of a 10 lb. weight which has fallen freely through 4 ft., its kinetic energy must be 10×4 ft.-lbs., or $10 \times 32{\cdot}2 \times 4$ ft.-poundals.

But we ought to be able to calculate the kinetic energy of a body in motion, however it acquired its speed; for example, if it was shot from a gun. We can do this by finding the amount of work it can do in losing its speed, instead of by calculating the amount of work which was done on it in order to give it its speed; in other words, by seeing how much energy the moving body can give out in coming to rest, instead of the amount of energy it absorbed in getting up speed.

66. Work done by a moving body in coming to rest. We will first find the value of the kinetic energy of a particular body, a carriage of mass 30 tons slipped from a train travelling at 60 miles an hour. The kinetic energy is by definition the work it can do in consequence of its motion; that is the work it will do before it is stopped. We shall see that the work done does not depend on the time it takes to stop it, so we can assume any time we please; 4 minutes is a likely time for the frictional resistance to pull it up without the help of the brakes, so we will take that, and we will assume that the frictional resistance is a constant force.

In 4 mins., or 240 secs., its speed is reduced from 88 ft. per sec. to zero; so the acceleration is $\frac{88}{240}$ ft. per sec. per sec. Since by the second law of motion the force is the product of the mass and the acceleration, the resisting force is $30 \times 2240 \times \frac{88}{240}$ poundals. Its initial speed is 88 ft. per sec. and its final speed is zero, so its average speed is $\frac{1}{2} \times 88$ ft. per sec., so in the 240 secs. it runs $\frac{1}{2} \times 88 \times 240$ ft.

Since the work it does in coming to rest is the product of the force and the distance moved, the work done in this case is $30 \times 2240 \times \frac{88}{240} \times \frac{1}{2} \times 88 \times 240$ ft.-poundals. This, then, was the kinetic energy of the carriage when it was slipped; it is changed into heat energy in the bearings, etc.

It will be seen that the 240, representing the time, cancels out, and the value of the kinetic energy depends only on the mass (30×2240 lbs.) and the speed (88 ft. per sec.), together with a numerical factor, $\frac{1}{2}$.

67. General formula for Kinetic Energy. We will next obtain in exactly the same way a general formula for the kinetic energy of a body of mass m (lbs. or grms.) moving at a speed of v (ft. per sec. or cm. per sec.). Assume that the body is brought to rest in t secs. by a constant force opposing its motion.

The acceleration is $\frac{v}{t}$ (ft. per sec. per sec., or cm. per sec. per sec.). By the second law of motion the force is the product of the mass and the acceleration, hence the constant resisting force must be $m \times \frac{v}{t}$ (poundals or dynes). The average speed of the body is $\frac{v}{2}$ (ft. per sec. or cm. per sec.), so in t secs. it moves $\frac{v}{2} \times t$ (ft. or cm.).

Since the work it does in coming to rest is the product of the force and the distance moved, the work done in this case is $m \times \frac{v}{t} \times \frac{v}{2} \times t$ or $\mathbf{\frac{1}{2}mv^2}$ (ft.-poundals or ergs). This then is the initial kinetic energy.

It should be noted here that the constant opposing force need not be the result of friction, as it was in the case of the slipped railway carriage; in that case the kinetic energy is turned into heat energy. It may be the constant attraction of the earth on the body itself, when the body is projected vertically upwards; in that case the kinetic energy is turned into potential energy. Or it may be the resistance opposed by a water wheel or turbine to the flow through it of water or steam; in that case the kinetic energy may be turned into electric energy, if the turbine is used to drive a dynamo.

It should also be noted that the formula $\frac{1}{2}mv^2$ gives a result in ft.-poundals or ergs according as the British or C. G. S. system of units is used, **but not in ft.-lbs.** This is a matter in which innumerable mistakes are made. It must be remembered that the "lb. wt." is not a dynamical unit and should be used only with the greatest caution in working out dynamical problems; in dealing with work it is often convenient to use that unit, and after the value of the kinetic energy has been calculated in ft-poundals it is therefore advisable to reduce it to ft.-lbs., by dividing the number of ft.-poundals by 32·2 (see Art. 43).

Ex. 3. A body of mass 12 lbs. moves at 6 ft. per sec. What is its kinetic energy in ft.-poundals? State its value in ft.-lbs.

It is brought to rest by a constant force in a distance of 6 ft.; what is the value of the force?

Ex. 4. A bullet weighs 1·75 grms. and moves at 300 metres per sec.; what is its kinetic energy? It acquired this speed while moving 15 cm. in the barrel of a pistol under a constant force; what was the force?

Ex. 5. An engine and train together weigh 200 tons, and acquire a speed of 60 miles an hour in 3 minutes from rest. What is the final kinetic energy? Neglecting frictional resistances, if the horse-power of the engine is constant, what is its value? If the tractive force is constant throughout, what is the greatest value of the H. P.?

The K. E. is $\frac{1}{2} \times 200 \times 2240 \times 88^2$ ft.-poundals or 1735 million ft.-poundals, or 53·9 million ft.-lbs. Hence the work done per min. is $\frac{1}{3} \times 53 \cdot 9$, or 17·97, million ft.-lbs.; hence the constant horse-power required is $\frac{17,970,000}{33,000}$ or **544** H. P.

But if the tractive force is constant throughout, the acceleration is constant, and equal to $\frac{88}{3 \times 60}$ ft. per sec. per sec. Hence the necessary tractive force is $200 \times 2240 \times \frac{88}{3 \times 60}$ poundals, or 6800 lbs. wt. When the speed is 60 miles an hour, work is done by this force at the rate of $6800 \times 1760 \times 3$ ft.-lbs. per min., and the H.P. is then **1088** H.P.; this is its greatest value, and it rises from 0 at the start to 1088 H. P. at full speed.

Ex. 6. If an aeroplane is loaded to a total of 1200 lbs., and the effective H. P. at the propeller is 44, how long will it take to attain a speed of 60 miles an hour from rest, if frictional resistance is neglected?

The kinetic energy at full speed can be calculated, and we know that in each minute the kinetic energy is increased by $44 \times 33,000$ ft.-lbs.

Ex. 7. A bullet of mass 1 oz. moving at 1500 ft. per sec., strikes a target and falls dead. Find the loss of kinetic energy, expressed in ft.-poundals and ft.-lbs.

***Ex. 8.** If 778 ft.-lbs. of work are equivalent to 1 British Thermal Unit (see Art. 63), calculate the number of British Thermal Units generated in Ex. 7.

2184 ft.-lbs. are equivalent to $\frac{2184}{778}$ or **2·8** B. TH. U. So the heat generated would raise the temperature of 1 oz. of water through $16 \times 2 \cdot 8°$ F. Since the specific heat of lead is ·0315, the heat generated would raise the temperature of 1 oz. of lead through $\frac{16 \times 2 \cdot 8}{\cdot 0315}$ or 1426° F. But lead melts at 635° F., and its latent heat of fusion is 5·37, so if the heat generated remains in the bullet, it will be completely melted by the blow.

***Ex. 9.** A steel shell weighing 12 lbs. moving at a speed of 1200 ft per sec. is brought to rest by striking a piece of armour plate; calculate the number of B. TH. U. which are generated. If all the heat is spent in heating the shell, what will be its rise in temperature, the specific heat of steel being ·118?

68. Kinetic Energy of a number of moving bodies. If several bodies are moving with different velocities in different directions, we can find the total kinetic energy of the whole group by calculating the kinetic energy of each body separately, and adding the results together. For the value of the kinetic energy of the whole group in ft.-poundals is the work which can be done before the whole comes to rest, and each body contributes its share to this total, whatever be its direction of motion. For the amount of heat or other form of energy generated when a certain amount of work is done does not depend on the direction of the displacement, but on the product of its magnitude and the acting force in the same direction.

69. Experimental verification of formula for kinetic energy. Fasten a long spiral spring to a hook at the back end of the plane as in Fig. 44; to the other end of the

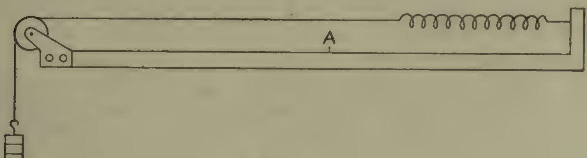


Fig. 44.

spring fasten a thread long enough to pass over the pulley, and attach weights to the thread until the spring has been stretched so that its end reaches to a marked point *A* exactly 5 inches from the end of the unstretched spring. In a particular case the weight required was ·85 lb. Then (see Art. 57) the average force during the extension was $\frac{1}{2} \times \cdot 85$ lb. wt. and the work done during extension (*i.e.* the potential energy stored in the stretched spring) is $\frac{1}{2} \times \cdot 85 \times \frac{5}{12}$ ft.-lbs. or $\frac{1}{2} \times \cdot 85 \times \frac{5}{12} \times 32 \cdot 2$ ft.-poundals, or **5·703** ft.-poundals.

Tilt the plane until a trolley will run down towards the spring with uniform speed; attach the thread to the trolley so that when the trolley is at the top of the plane the spring is stretched to the point *A*. Arrange a vibrating spring making a known number of vibrations a second so that you can measure the speed of the trolley after the spring has contracted to its unstretched length. Hold the trolley so that the end of the spring is exactly at *A*; release it, and determine its speed after the spring has ceased to act on it. The speed was found to be 1·3 ft. per sec., and its mass was 6·8 lbs. The formula for the kinetic energy gives $\frac{1}{2} \times 6{\cdot}8 \times 1{\cdot}3^2$ ft.-poundals, or **5·745** ft.-poundals.

Now the potential energy of the spring has disappeared, and the kinetic energy of the trolley has appeared; there can be no transformation into other forms of energy, since the friction was counterbalanced; hence the kinetic energy of the trolley must be equal to the potential energy of the spring. So the expression $\frac{1}{2}mv^2$ ft.-poundals gives in this case an approximately correct value for the kinetic energy of a body of mass *m* lbs. moving with a speed of *v* ft. per sec.

70. Use of kinetic energy in solving problems. The law of conservation of energy furnishes us with a short cut to the solution of many dynamical problems, if we do not want to know the position of the bodies at every instant throughout the motion. If we know that there is no loss of mechanical energy, i.e. that no other form of energy is produced during the motion, the sum of the potential and kinetic energies must be constant. We will take various examples of this.

(i) Suppose a man stands on a cliff 100 ft. above the sea, and throws a stone in any direction with a speed of 50 ft. per sec. What will be its speed on striking the sea, if the resistance of the air is neglected? If the stone has a mass of *m* lbs., its kinetic energy at first is $\frac{1}{2}m\,50^2$ ft.-poundals. On reaching sea-level it has lost $m \times 32{\cdot}2 \times 100$ ft.-poundals of potential energy, so its total kinetic energy is $\frac{1}{2}m\,50^2 + m \times 32{\cdot}2 \times 100$ ft.-poundals. If its speed is v ft. per sec., its kinetic energy is $\frac{1}{2}mv^2$ ft.-poundals, so v can be calculated from the equation

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m\,50^2 + m \times 32{\cdot}2 \times 100.$$

Ex. 10. A stone whose mass is 1 lb. is thrown vertically upwards with a speed of 50 ft. per sec. How high will it rise before all its kinetic energy is converted into potential energy?

(ii) Let a trolley of mass M lbs. stand on a plane sloped to balance frictional resistances; let a weight of m lbs. be attached to it by a string passing over a pulley. Let them start from rest; call their velocity v ft. per sec. after they have moved s ft. Then the loss of potential energy in the weight is mgs ft.-poundals, and the gain of kinetic energy is $\frac{1}{2}Mv^2$ ft.-poundals in the trolley and $\frac{1}{2}mv^2$ ft.-poundals in the weight, or a total gain of $\frac{1}{2}(M+m)v^2$ ft.-poundals.

Hence $$\tfrac{1}{2}(M+m)v^2 = mgs.$$

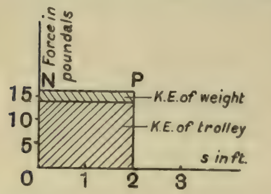
Owing to the downward slope of the plane, the trolley has lost some potential energy, but this is turned into heat energy through friction, so neither appears in the above equation.

Ex. 11. A mass of 1 lb. is attached to a trolley of mass 6 lbs.; work out as above the speed when the trolley has moved 18 inches.

***Ex. 12.** Call the tension of the string in the above arrangement T poundals. Then the work which this force does on the trolley is Ts ft.-poundals. Hence $Ts = \frac{1}{2}Mv^2$. Substituting the value of $\frac{1}{2}v^2$ from the above equation, we get $Ts = M\dfrac{mgs}{M+m}$ or $T = \dfrac{Mm}{M+m}\cdot g$ poundals.

***Ex. 13.** Find in lbs. wt. as in Ex. 12 the tensions of the string by which (i) a 1 lb. weight, (ii) a $\frac{1}{2}$ lb. weight, pulls a 4 lb. trolley.

The change of energy in this and similar cases can be represented graphically, as follows. Draw the force-displacement curve, taking as abscissae the displacements, and as ordinates the acting forces; then (see Art. 55) the area bounded by this curve, the ordinates at its ends and the horizontal axis represent the work done during the displacement. This area will therefore represent the gain or loss of energy. Take as an example the case of a $\frac{1}{2}$ lb. weight attached to a 4 lb. trolley. The force of gravity is $\frac{1}{2} \times 32{\cdot}2$ or $16{\cdot}1$ poundals, so the work done by gravity in a displacement of 2 ft. is $16{\cdot}1 \times 2$ ft.-poundals, and this is represented in Fig. 45 by the area $O2PN$, measured in ft.-poundals. Hence the area $O2PN$ represents also the loss of potential energy of the weight, and, therefore, the total kinetic energy of the trolley and weight acquired in running a distance represented by $O2$. Since the speeds of the trolley and weight are the same, this total kinetic energy is shared between them in proportion to their masses. A horizontal line through this area is drawn to divide it into two parts in the proportion 4 to $\frac{1}{2}$, and hence these parts represent the kinetic energies of the trolley and weight respectively.



Scale:
1 in. to 4 ft.
1 in. to 40 poundals.
1 sq. in. to 160 ft.-poundals.

Fig. 45.

(iii) Suppose that two weights, A of m_1 lbs. and B of m_2 lbs. (of which m_2 is the greater), are connected by a light cord and hung over a light frictionless pulley as in Fig. 46. Suppose that they have moved a distance of s ft. under the action of gravity, and that they have acquired a speed of v ft. per sec.; A will of course be moving upwards and B downwards, and their speeds

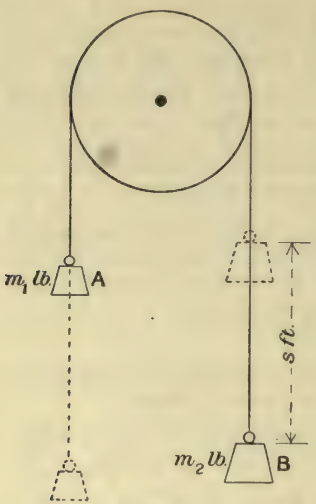


Fig. 46.

must be equal since they are connected by a cord. A has gained potential energy to the extent of m_1gs ft.-poundals, and B has lost potential energy to the extent of m_2gs ft.-poundals. A has gained kinetic energy to the extent of $\frac{1}{2}m_1v^2$ ft.-poundals, and B has gained kinetic energy to the extent of $\frac{1}{2}m_2v^2$ ft.-poundals. Considering the two bodies together as one "system," there can be no change in the total energy (since none has been transformed into heat, etc.); hence the gain must on the whole equal the loss.

Hence $$m_1gs + \tfrac{1}{2}m_1v^2 + \tfrac{1}{2}m_2v^2 = m_2gs,$$

or $$\tfrac{1}{2}(m_1 + m_2)v^2 = (m_2 - m_1)gs.$$

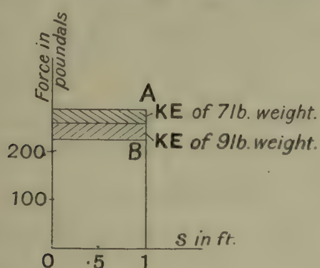
From this equation the velocity at any point can be calculated.

Ex. 14. If the two weights are 14 lbs. and 15 lbs., find as above their speed when they have each moved through 2 ft.

***Ex. 15.** Call the tension of the string in the above arrangement T poundals. Then the work which this force does on A is Ts ft.-poundals. The gain in energy of A is $m_1gs + \frac{1}{2}m_1v^2$ ft.-poundals. Substituting the value of v^2 from the above equation we get $T = \frac{2m_1m_2}{m_1+m_2} \cdot g$ poundals.

***Ex. 16.** Work out as in Ex. 15 the tension of the string connecting weights of 7 and 10 lbs.; express the result in lbs. wt.

Fig. 47 represents these changes of energy, graphically as before, in a case where A has a mass of 7 lbs. and B a mass of



Scale:
1 in. to 2 ft.
1 in. to 400 poundals.
1 sq. in. to 800 ft.-poundals.

Fig. 47.

9 lbs., after moving through 1 ft. from rest. The rectangle OA

represents the loss of P. E. of the larger weight, and rectangle OB represents the gain of P. E. of the smaller weight. Hence the remainder of OA represents the K. E. acquired by the two weights. As before this area is divided into two parts to show the kinetic energies of the two weights separately, by dividing AB in the ratio 7 to 9.

*(iv) Suppose we have a "simple pendulum" consisting of a "bob" of mass ·2 lb. hung by a string of length 3 ft.; assume that the string is so light that its weight may be neglected, that the bob is so small that we can treat it as if its whole mass was concentrated at its centre of gravity, and that we may neglect the frictional resistance of the air to the motion. If the bob is drawn aside until the string makes an angle of 30° with the vertical, it will then be at a vertical distance below its point of support of 3 cos 30°, or 2·60 ft.

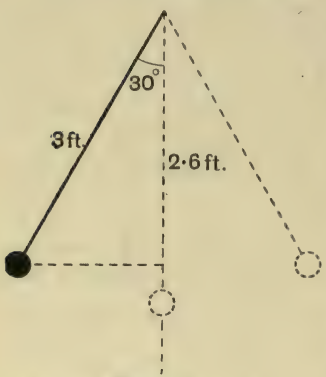


Fig. 48.

Hence it has been raised through 3·0 – 2·60 or ·40 ft. from its lowest position (when the string was vertical), so its potential energy is ·2 × 32·2 × ·4 or 2·58 ft.-poundals. If the bob is now released, it will swing to its lowest position, and the potential energy will be changed into kinetic energy; if v ft. per sec. is its velocity as it passes through this point, its kinetic energy is $\frac{1}{2} \times \cdot 2 \times v^2$ ft.-poundals, so by the law of conservation of energy

$$\tfrac{1}{2} \times \cdot 2 \times v^2 = 2{\cdot}58 \text{ ft.-poundals.}$$

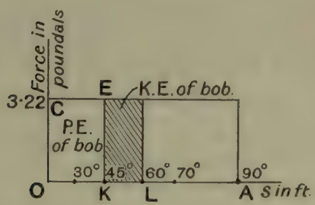
Hence the value of v may be calculated; and in a similar way its speed at any point may be found. (Note that this method does not give us the speed or position at any *instant.*)

The same law shows that the kinetic energy at its lowest point will produce the original amount of potential energy when the bob again comes to rest; i.e. it rises to the same height on the other side of the centre line.

Ex. 17. Find the kinetic energy and hence the speed at the lowest point, when the string was inclined at 10° to the vertical at starting.

Ex. 18. During the swing the string encounters a fixed peg 1 ft. vertically below the point to which the string is attached; draw a diagram to show the position of the bob when at its highest point.

The changes of energy can be shown on a diagram as follows. Take the case of a bob of mass ·1 lb. on the end of a string 1 ft. long. On the force displacement diagram (Fig. 49) take OA to represent 1 ft., the vertical displacement when the string



Scale: 1 in. to 1 ft.
1 in. to 8 poundals.
1 sq. in. to 8 ft.-poundals.

Fig. 49.

is inclined 90° to the vertical, and mark off distances along OA to represent on the same scale the vertical heights above the lowest point for different inclinations of the string; these lengths can be taken from a table of natural cosines. For example, for an inclination of 45° the length OK will be $1 - \cos 45°$ or ·2929 ft.

Take on the force axis the point C representing $\cdot 1 \times 32{\cdot}2$ poundals; then the rectangle AC will represent the potential energy of the bob when the string is horizontal, and the rectangle KC gives its potential energy when the string is inclined at 45°, and so on. If we start the bob from rest with the string at an angle of, say, 60° (i.e. at a vertical height OL above the lowest point) its potential energy is represented by the rectangle

LC. When the bob has dropped till the string has an inclination of 45°, the potential energy will be given by the rectangle *KC*, and therefore the loss of P.E. will be given by rectangle *LE*, so the kinetic energy at this point of its swing is given by rectangle *LE*.

On the scale of Fig. 49, the length *KL* is ·207 in., and *KE* is ·4025 in., so the area *LE* is ·0833 in., which represents ·666 ft.-poundals. Hence if v ft. per sec. be the speed of the bob when the string is inclined at 45° we have $\frac{1}{2} \times \cdot 1 \times v^2 = \cdot 666$, or $v =$ **3·65** ft. per sec.

In a similar way we can find the kinetic energy (and hence the speed) at any point of the swing, whatever the length of the string or the height from which the swing started.

*(v) Take a spiral spring, and suspend it from a fixed point. Let its lower end when unstretched stand at *A*. Hang a weight of m lbs. on it; when it has come to rest let the lower end of the spring stand at *O*. Denote the distance *AO* by a ft. The tension of the spring in this state is mg poundals. We know that in a spiral spring the extension is proportional to the tension, so for any other position of the bottom of the spring, x ft. below *A*, the force it will exert will be $\frac{x}{a} mg$ poundals.

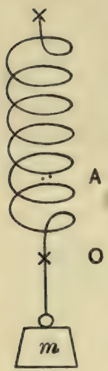
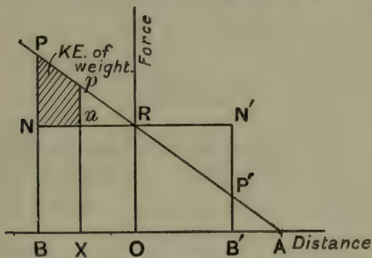


Fig. 50.

Draw the force-distance curve *RA* for the spring as in Fig. 51, which is drawn for a particular case but of course illustrates the general case. Then the area of the triangle *ORA* represents the potential energy stored in the spring when its bottom end is at *O*.

Now pull the weight down through a distance b ft. (less than a), so that the lower end of the spring comes to *B*; let *OB* represent b ft. In this operation an additional amount of potential energy is stored in the spring, represented by the area *ORPB*, and the potential energy of the weight is decreased by an amount represented by the area *ORNB* (since *OR* represents the attraction of the earth on the weight); hence the work done in pulling it down is represented by the area *RNP*.

If the weight is now released, it will rise because the tension of the spring (which is now *PB*) is greater than the weight; when it reaches a point *X*, the weight will have acquired potential energy represented by the area *NX*, and the spring will have given out potential energy represented by the area *pXBP*. The difference between these (i.e. the area *pnNP*) must therefore have taken the form of kinetic energy in the weight; hence we can calculate the velocity (v ft. per sec.) of the weight at any



Scale:
1 in. to ·8 ft.
1 in. to 400 poundals.
1 sq. in. to 320 ft.-poundals.

Fig. 51.

given point of its upward path, since we know the value of $\frac{1}{2}mv^2$. When it reaches *O* the weight will have kinetic energy represented by the area *PRN*, so it will rise beyond this point and will continue to rise until this kinetic energy has been converted into potential energy again. Suppose this happens when it reaches the point *B′*. Then, since passing *O* the potential energy of the weight has increased by the area *RN′B′O*, and the potential energy of the spring has decreased by the area *RP′B′O*; hence there has been an increase of potential energy represented by the area *RN′P′*. Hence *RN′P′* is equal in area to *PRN*; since these triangles are similar, $RN' = RN$. Hence the weight will rise as far above *O* as it started below it, before losing its kinetic energy; it will then fall again, and oscillate about the equilibrium position *O*. These results can easily be verified by experiment.

Ex. 19. For a mass of 7 lbs., a is found to be ·6 ft. If b is ·4 ft., find the kinetic energy of the weight as it passes O, and hence its velocity, using Fig. 51 (which is drawn for this case), or one to a much larger scale.

Ex. 20. In above case, find the kinetic energy of the weight, and hence its velocity, when it is ·3 ft. from O.

It will be convenient for later use to calculate the resultant force on the weight, as well as its velocity, at any point X of its path. The resultant force is the difference of the upward pull of the spring (represented by pX) and the weight (or nX); and by similar triangles $\frac{pn}{nR} = \frac{OR}{OA}$. If we denote the distance OX of the weight from O by y ft., this gives

$$\text{resultant force} = y \times \frac{mg}{a} \text{ poundals.}$$

Hence the force on the weight is proportional to its distance from O; and it is always directed towards O, since at points above O the pull of the spring is less than the weight.

Ex. 21. For the case of Ex. 19, determine the force on the weight when it is (a) ·4 ft. below O, (b) ·1 ft. below O and (c) ·4 ft. above O.

71. Problems involving unmechanical forms of energy. Many problems in which there is a transformation of kinetic or potential energy into heat or some other form of energy can be solved by applying the law of conservation of energy. For example, take the case of a man riding a bicycle at 5 miles an hour, who starts free-wheeling down a slope of 1 in 20, 200 yards long; suppose that he and his bicycle together weigh 180 lbs., and that the frictional resistance to motion is 7 lbs. wt. We can find his speed at the bottom as follows. He descends a vertical height of $\frac{1}{20} \times 200$ yds. or 30 ft., so he loses potential energy to the extent of $180 \times 32{\cdot}2 \times 30$ ft.-poundals. His initial speed is $\frac{5}{60} \times 88$ or $7{\cdot}33$ ft. per sec., so his initial kinetic energy is $\frac{1}{2} \times 180 \times 7{\cdot}33^2$ ft.-poundals. The work done against the frictional resistance (which is $7 \times 32{\cdot}2$ poundals) in travelling 200×3 ft. is $7 \times 32{\cdot}2 \times 200 \times 3$ ft.-poundals. Hence the kinetic energy at the bottom is

$$\tfrac{1}{2} \times 180 \times 7{\cdot}33^2 + 180 \times 32{\cdot}2 \times 30 - 7 \times 32{\cdot}2 \times 200 \times 3 \text{ ft.-poundals}$$

or 43,440 ft.-poundals. If his speed there is v ft. per sec., his kinetic energy is $\frac{1}{2} \times 180 \times v^2$ ft.-poundals; hence

$$\tfrac{1}{2} \times 180 \times v^2 = 43{,}440 \text{ or } v = 21{\cdot}97 \text{ ft. per sec.}$$

or 15 miles an hour, about.

Ex. 22. How far will this bicycle now run up a slope of 1 in 30? Call the distance x feet; then the vertical rise is $\frac{x}{30}$ ft., so the gain of potential energy is $180 \times 32{\cdot}2 \times \frac{x}{30}$ ft.-poundals. The work done against friction is $7 \times 32{\cdot}2 \times x$ ft.-poundals.

Hence $$43{,}440 = 180 \times 32{\cdot}2 \times \frac{x}{30} + 7 \times 32{\cdot}2 \times x.$$

Ex. 23. The car and passengers of a switch-back weigh 15 cwt., the frictional resistance is 28 lbs. wt. The car starts from rest at a height of 20 ft., descends a slope of 1 in 5 to ground level, then runs up a slope of 1 in 10. Find the speed at ground level, and the distance it will run up the slope before coming to rest.

Ex. 24. A "6-inch" gun weighs 7 tons 8 cwt., and after it is fired it begins to recoil at a speed of 15 ft. per sec. Find its kinetic energy. The recoil is stopped in 13·3 inches by a constant force; find its magnitude.

Its kinetic energy is 1·865 million ft.-poundals. If x poundals is the frictional resistance, the work done by this kinetic energy is $x \times \frac{13{\cdot}3}{12}$ ft.-poundals, from which we get $x = 1{\cdot}683$ million poundals, or **23·33 tons wt.**

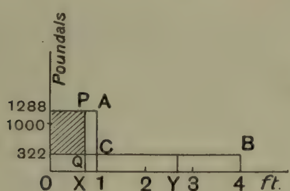
Ex. 25. A 12-inch gun weighs 50 tons, and after it is fired it begins to recoil at a speed of 18·42 ft. per sec. Find its kinetic energy. The recoil is stopped in 18 inches by a constant force; find its magnitude.

72. Storage of Energy. Energy is more valuable than any material in the world; without it the world would be dead. Human beings have passed through a Stone Age, a Bronze Age, etc., when they could avail themselves only of the energy in themselves or animals, and could effect comparatively little. This is an Age of Energy, in which we utilise the energy stored in coal (which we are using up many thousand times faster than it was accumulated and so are "living on our capital"), and to a small extent we utilise the energy of waterfalls (which is a case of "living on income"). When we have extracted the energy from coal, it is often convenient to store it temporarily, in order to meet fluctuations in the demand or supply; for example, an electric tramway makes a very variable demand on the output of power, for each tramcar calls for a very rapid expenditure of energy when it is started, and returns some energy to the mains when it is stopped. To ensure enough power to start several cars at once would demand a very large engine, which would often be working far below its maximum output, and therefore inefficiently; so a smaller engine is used, which transforms energy at a steady rate and this energy is stored (in batteries of accumulators) to meet exceptional demands. In the same way, the human body stores for a few hours the energy derived from food, which can be used as required.

Again, consider a single cylinder petrol motor driving a motor car or motor bicycle; it is only during one stroke out of four that it is doing work, and part of the energy it draws from the petrol has to be stored during the working stroke in order to provide for the demand during the next three strokes. This can be done by the car or bicycle itself; it acquires enough kinetic energy during the working stroke to overcome the frictional resistance to its motion during the next three strokes. But the result is a jerky kind of motion, for the accumulation of kinetic energy in the car means an increase of speed, which decreases again as the store of kinetic energy is drawn upon, so that every working stroke of the engine would be noticeable. This is actually the case in paddle steamers, where the engines generally make comparatively few strokes a minute; at the beginning and end of these strokes the engine is working at a disadvantage (see Art. 47) and the stored kinetic energy of the ship is drawn upon, so that it is quite possible to perceive the alternate acceleration and retardation.

Ex. 26. Consider a single cylinder motor cycle, weighing with the rider 400 lbs., going at 40 ft. per sec. (about 27 miles an hour); and suppose the frictional resistance of the road, wind, etc., is 10 lbs. wt., and that at the given speed the engine makes 2400 strokes per minute. Suppose for simplicity that the rider is rigidly attached to the cycle, forming one body with only a simple motion of translation forwards.

In 1 sec. the bicycle moves 40 ft. and makes 40 strokes, hence it moves 1 ft. during a stroke. In moving 1 ft. it does $10 \times 32{\cdot}2 \times 1$ or 322 ft.-poundals. So in moving 4 ft. it does 1288 ft.-poundals; and all this energy is provided by the engine in one working stroke, since only one stroke out of four is a working stroke. During the working stroke 322 ft.-poundals will be used up, so the balance, or $1288 - 322$, or 966 ft.-poundals is stored up to be expended in the remaining 3 ft. before the next working stroke begins. This may be represented as in Fig. 52, O is the position at the beginning of a working stroke, the area OA represents the 1288 ft.-poundals provided by the engine in a working stroke (assuming that the engine provides it steadily throughout the stroke) and OB represents the energy spent during the 4 ft. run. At any point X the work done by the engine is represented by OP, and the work spent by OQ, hence the kinetic energy added is represented by the shaded area. At the end of the working stroke the addition of kinetic energy reaches 966 ft.-poundals, as we have seen. At a later point Y, this will have been reduced by the work represented by rectangle CY, and after 4 ft. from O it will all have been expended.



Scale:
1 in. to 4 ft.
1 in. to 4000 poundals.
1 sq. in. to 16000 ft.-poundals.

Fig. 52.

Therefore the kinetic energy of rider and bicycle will vary by this amount, 966 ft.-poundals. Now, if we neglect the kinetic energy of the separate moving parts, wheels, etc., and regard only the forward speed of the whole, the average kinetic energy is $\frac{1}{2} \times 400 \times 40^2$ or 320,000 ft.-poundals; so the variation is roughly 1000 in 300,000 or $\frac{1}{3}\,{}^{\circ}/_{\circ}$.

It is worth while examining this a little more closely, to find out whether this variation will be noticeable. We may use round numbers for this purpose, and assume that the kinetic energy varies between 319,500 and

320,500 ft.-poundals; then the speed will vary between $\sqrt{\frac{2 \times 319,500}{400}}$ and $\sqrt{\frac{2 \times 320,500}{400}}$, i.e. between 39·97 and 40·03 ft. per sec. Hence in a working stroke (i.e. $\frac{1}{40}$ sec.) the speed increases by ·06 ft. per sec., so the acceleration is ·06 × 40 or 2·4 ft. per sec. per sec. If the rider himself weighs 11 stone, the force on him needed to produce this acceleration is 154 × 2·4 poundals, or 11·5 lbs. wt.

Hence the bicycle thrusts him forward ten times a second with a force of about 11 lbs. wt. This would be intolerable, but as a matter of practice the blows are softened by springs, and his own speed consequently keeps much more constant. However, as he supplies part of the inertia of the combined bicycle and rider, he cannot avoid shocks if this inertia is to be the storehouse of energy; he is more comfortable if a flywheel is fitted to the engine to serve as a storehouse.

73. Flywheels. In the case of stationary engines driving machinery there may be no large mass moving at a high speed, as with trains or paddle steamers or bicycles, which can store energy to provide for the times when the engine is not doing effective work. For this purpose, a massive flywheel is fitted to the shaft of the engine; when this is rotating at high speed it contains a large amount of kinetic energy.

A form of flywheel that is theoretically the simplest consists of a single circular ring supported on an axle by light spokes, like a bicycle wheel with a solid tyre made of cast iron or lead. If we can assume that the cross-section of this tyre is so small that we can regard it as a thin wire lying along the mean circumference of the rim, and if we know the diameter and weight and rate of revolution of the wheel, we can easily calculate the kinetic energy it contains. Suppose for example that the radius is 14 inches and that it makes 10 revolutions a second and that the rim weighs 100 lbs. Then every point of the rim moves through the length of the circumference (i.e. $2\pi \times 14$ inches or 7·33 ft.) in one-tenth of a second; so its speed is 7·33 × 10 ft. per sec., or 73·3 ft. per sec. Hence we have a mass of 100 lbs. moving at a speed of 73·3 ft. per sec.; so its kinetic energy is $\frac{1}{2} \times 100 \times 73{\cdot}3^2$ ft.-poundals or 268,600 ft.-poundals.

In the more general case, if the mass of the rim is m lbs., and its radius is r ft., and the wheel makes n revolutions per second, a point on the rim moves $n \times 2\pi r$ ft. in a second, so its kinetic energy is $\frac{1}{2} mn^2 4\pi^2 r^2$ ft.-poundals.

In practice, flywheels are not made in this simple form, but it is always possible to determine, either by calculation or experiment, the radius of the equivalent simple flywheel which has the same total mass, and which contains the same kinetic energy when it revolves at the same number of revolutions a minute. For example, suppose that the flywheel is a circular disc like a grindstone; when it revolves, the parts near the axle are moving slowly, and those on the outer edge are moving rapidly, so that a pound of the material has a different kinetic energy according to its distance from the axle. But there must be an average radius at which all the material might be collected and yet have the same total kinetic energy, the decrease of speed of the parts moved nearer to the axle balancing the increase of speed of those moved further from the axle.

As an illustration of this, consider a flywheel formed, as in Fig. 53, of two hoops of circumference 3 and 6 ft. respectively, made of iron rod which weighs 2 lbs. to the foot; the spokes are supposed to be so light that we can neglect their kinetic energy. If the wheel turns n times a second, the speeds of a point on the hoops are $3n$ and $6n$ ft. per sec. respectively. The masses are 3×2 and 6×2 lbs., so the total kinetic energy of the wheel is $\frac{1}{2} \times 6 \times (3n)^2 + \frac{1}{2} \times 12 \times (6n)^2$ ft.-poundals. The "equivalent simple flywheel" must have the same mass (18 lbs.) and its radius (R ft.) must be such that it has the same kinetic energy when it turns at the same speed (n per sec.). Its circumference will be $2\pi R$ ft.; hence the speed of a point on it will be $2\pi R \times n$ ft. per sec.; so its kinetic energy will be $\frac{1}{2} \times 18 \times (2\pi Rn)^2$ ft.-poundals.

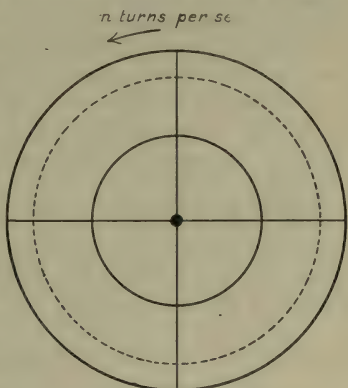


Fig. 53.

Hence $\frac{1}{2} \times 18 \times 4\pi^2 R^2 n^2 = \frac{1}{2} \times 6 \times 9n^2 + \frac{1}{2} 12 \times 36n^2$.

$$\therefore\ R = \cdot 827 \text{ ft.}$$

Note that this gives a circumference of 5·195 ft. (shown dotted in Fig. 53), which is intermediate between the lengths of the two hoops, but *not* the mean of the two; also that the simple flywheel is equivalent to the other for all speeds, since n disappeared from the equation for finding the radius.

74. Radius of Gyration. A special name is given to the radius of this equivalent simple flywheel; it is called the *Radius of Gyration* of the flywheel. It may be defined as follows: "the radius of gyration of a flywheel is the distance from the axle at which the whole mass must be collected to form a simple flywheel having the same kinetic energy when making the same number of revolutions per second."

The value of the radius of gyration of a given flywheel can be calculated mathematically in some cases, or can be determined by experiment, as will be shown presently. When the mass also is known, the kinetic energy for a given speed of rotation can readily be calculated.

Ex. 27. A flywheel has a mass of 5 tons and a radius of gyration of 3 ft. Calculate its kinetic energy when turning at 80 revs. per min.

The speed of a mass collected at the end of the radius of gyration is $2\pi \times 3 \times 80$ ft. per min. or 25·13 ft. per sec. Hence the kinetic energy is $\frac{1}{2} \times 5 \times 2240 \times 25{\cdot}13^2$ ft.-poundals.

Ex. 28. A flywheel has a mass of 12 lbs. and a radius of gyration of 4 inches. Calculate its kinetic energy when turning 1200 revs. per minute.

Disc flywheel. If the flywheel consists of a uniformly thick circular disc like a grindstone, it can be shown that there is a very simple connection between its radius of gyration and the radius of the disc; if r be the radius of the disc, the radius of gyration is $\frac{r}{\sqrt{2}}$. Integral calculus is needed to prove that this result is exactly true, but it can be verified roughly by imagining the disc cut into a series of rings and calculating their kinetic energies as was done for the two hoops in Art. 73.

For example, take a disc flywheel of diameter 2 ft.; imagine a series of concentric circles, 2 ins. apart, to be drawn on the face, dividing it into 6 rings. Call the thickness of the disc d ins., its speed n revolutions a second and let 1 cub. in. of the material weigh m lbs. These values do not affect the radius of gyration, but will be used in finding it. The mean radius of the rings will be 11, 9, 7, ... 1 in.; so the mean speeds of a point on each will be $2\pi n \,.\, 11$, $2\pi n \,.\, 9$, etc. ins. per sec.; the area of the largest will be $\pi \,.\, 12^2 - \pi \,.\, 10^2$ or 44π sq. ins.; the area of the next, $\pi \,.\, 10^2 - \pi \,.\, 8^2$ or 36π sq. ins.; and so on. Hence the volume of the largest will be $44\pi d$ cub. ins., and its mass $44\pi dm$ lbs., and so on. Hence the kinetic energy of the largest will be, approximately, $\frac{1}{2} \times 44\pi dm \left(\frac{2\pi n \,.\, 11}{12}\right)^2$ ft.-poundals, or $\frac{\pi^3}{72} dmn^2 \times 44 \times 11^2$; the kinetic energy of the next will be $\frac{\pi^3}{72} dmn^2 \times 36 \times 9^2$; and so on.

Hence the total kinetic energy will be approximately

$$\frac{\pi^3}{72} dmn^2 (44 \times 11^2 + 36 \times 9^2 + \ldots)$$

or

$$\frac{4\pi^3}{72} dmn^2 (11^3 + 9^3 + \ldots).$$

The whole volume of the wheel is $\pi \times 12^2 d$ cub. ins., so its mass is $\pi \,.\, 12^2 dm$ lbs. If we denote the radius of gyration by R ft., the speed of this mass collected at a distance R ft. from the axle will be $2\pi Rn$ ft. per sec. So the kinetic energy of the wheel is $\frac{1}{2} \times \pi \,.\, 12^2 dm \,(2\pi Rn)^2$ ft.-poundals.

Hence $$\tfrac{1}{2}\pi \,.\, 12^2 dm \,.\, 4\pi^2 R^2 n^2 = \frac{\pi^3}{18} \,.\, dmn^2 (11^3 + 9^3 + \ldots)$$

or $$R^2 = \cdot 493 \quad \text{and} \quad R = \mathbf{\cdot 702} \text{ ft.}$$

Now the formula gives $\frac{1}{\sqrt{2}}$ ft. for R, which is **·707** ft., so the approximation is within 1 %.

Ex. 29. Find the radius of gyration of a disc flywheel made of cast iron weighing ·26 lb. to the cubic inch, the diameter being 9 ins. and the thickness 2 ins.

Ex. 30. Find the energy stored in the above flywheel when turning at 1000 revs. per min.

Ex. 31. Find the energy stored in a grindstone of diameter 4 ft., 5 ins. thick, made of stone weighing 160 lbs. to the cub. ft., and turning at 30 revs. per min.

75. Measurement of radius of gyration by experiment. The flywheel should be mounted to turn freely on a horizontal axle; we will first suppose that the bearing friction is so small that it may be neglected. A weight is hung from a cord which is coiled round the axle, a loop on the end of the cord being put over a peg in the axle so that when the cord is uncoiled the weight is released. The weight is allowed to pull the flywheel round for a measured drop, and the speed of the flywheel, in turns per sec., is then measured. If we now measure the diameter of the axle we can calculate the speed, and hence the kinetic energy, of the weight. The loss of potential energy of the weight can be calculated, and hence the energy given to the flywheel. From this, and the known speed of rotation and mass of the flywheel we can deduce its radius of gyration. For example, a weight of 4 lbs. hangs by a cord coiled round the axle (whose diameter is 2 ins.); it starts from rest and is released when it has fallen 2 ft. The flywheel is then found to be revolving at 5 revs. per sec. The mass of the flywheel is known to be 3 lbs.

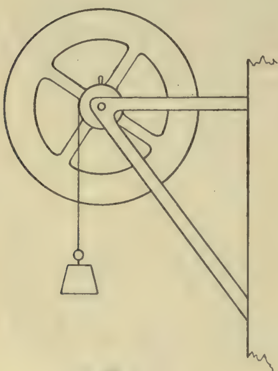


Fig. 54.

A point on the axle, at the instant when the weight is released, moves through $\pi \times 2$ inches in $\frac{1}{5}$ sec.; so the speed of the weight is then $\dfrac{\pi \times 2 \times 5}{12}$ ft. per sec. or 2·62 ft. per sec. Hence its kinetic energy is $\frac{1}{2} \times 4 \times 2{\cdot}62^2$ or 13·7 ft.-poundals. The loss of potential energy of the weight is $4 \times 32{\cdot}2 \times 2$ ft.-poundals, or 257·6 ft.-poundals; hence the energy given to the flywheel is $257{\cdot}6 - 13{\cdot}7$ or 243·9 ft.-poundals. If R ft. be its radius of gyration, the speed of the 3 lb. mass collected there would be $2\pi R \times 5$ ft. per sec., so its kinetic energy would be $\frac{1}{2} \times 3 \times (10\pi R)^2$.

Hence to find R we have the equation

$$150\pi^2 R^2 = 243 \cdot 9,$$

whence $$R = \cdot 406 \text{ ft. or } \mathbf{4 \cdot 87} \text{ inches.}$$

If the friction is large enough to make it necessary to allow for its effects, it can be done as follows. Suppose that a weight of ·4 lb. hung by a cord coiled round the axle is found to be just sufficient to keep the flywheel turning at a steady rate. Then during the 2 ft. drop of the 4 lb. weight work to the extent of $\cdot 4 \times 32 \cdot 2 \times 2$ ft.-poundals is spent in overcoming friction, and this amount must be subtracted from the energy (243·9 ft.-poundals) which we calculated was transferred to the flywheel. The calculation of the radius of gyration then proceeds as before.

Ex. 32. A flywheel weighs 3 tons; its axle has a diameter of 5 inches. It is found that a weight of 90 lbs. hung by a cord coiled round the axle just keeps it turning at a steady rate. If a weight of 5 cwt. is hung by the cord and released after it has descended 8 ft., the flywheel is observed to turn at the rate of one revolution in 3 secs. Find its radius of gyration.

76. Fluctuation of speed of an engine. Consider the case of a gas engine driving a dynamo, which requires 100 H.P., at an average of 300 revs. per min. Suppose that the gas engine supplies energy at a uniform rate during one stroke, and supplies none during the next (there being two strokes to each revolution); and suppose it is required that the speed should not rise above 305, nor fall below 295, revolutions a minute. How large must the flywheel be to keep the speed within these limits? The dynamo requires $100 \times 33{,}000$ or 3,300,000 ft.-lbs. in a minute, i.e. in 300 revolutions; so it requires 11,000 ft.-lbs. in one revolution or 5500 ft.-lbs. during one stroke. Hence during a working stroke the engine must produce 11,000 ft.-lbs. of energy, half of which is taken by the dynamo during the stroke, the other half being stored in the flywheel. Therefore during the working stroke the flywheel absorbs 5500 ft.-lbs. or

$5500 \times 32 \cdot 2$ ft.-poundals, and its speed increases from 295 to 305 revs. per min.; during the next stroke it gives out this energy and its speed drops again to 295 revs. per min.

If its mass is M lbs. and its radius of gyration is R ft., its kinetic energy at 305 revs. per min. (or $\frac{305}{60}$ revs. per sec.) will as before be $\frac{1}{2}M \times 4\pi^2 R^2 \cdot \frac{305^2}{60^2}$ ft.-poundals. And its kinetic energy at 295 revs. per min. is $\frac{1}{2}M \times 4\pi^2 R^2 \frac{295^2}{60^2}$ ft.-poundals. The difference between these two is $\frac{1}{2}M \times 4\pi^2 \frac{R^2}{60^2}(305^2 - 295^2)$ ft.-poundals.

Hence $$2\pi^2 \frac{MR^2}{60^2}(305^2 - 295^2) = 5500 \times 32 \cdot 2.$$

Or $$MR^2 = 5381 \text{ ft.-lb. units.}$$

If we decide that 5 cwt. would be a convenient weight for the flywheel, then $R^2 = \frac{5381}{5 \times 112}$ and $R = \mathbf{3 \cdot 1}$ ft.

Miscellaneous Exercises.

Ex. 1. In a boiler and engine which delivers 100 brake horse power, it is found that 10 per cent. of the heat generated by combustion of the coal is converted into mechanical energy. Each pound of coal burnt generates 14,800 B.T.U.; Joule's equivalent may be taken as 778. Calculate (i) the mechanical energy produced for each pound of coal; (ii) the number of ft.-lbs. of work done by the engine in an hour; and hence (iii) the number of lbs. of coal burnt per hour.

Ex. 2. A steel shell travelling at 2000 ft. per sec. is suddenly stopped; calculate its rise in temperature, if its specific heat is ·1, and if all the heat generated is retained in the shell.

Ex. 3. A three ton truck runs down a slope of 1 in 20 and gains a speed of 37 ft. per sec. in 500 ft. Find the retarding force of friction. What would have been its speed if there had been no friction?

Ex. 4. A shaft, 4 inches diameter, supported in two bearings, makes 80 revs. a min. The weight of the shaft and load on it amount to 4 tons, and the coefficient of friction is ·06. Find the H.P. absorbed in the bearings.

Ex. 5. In Ex. 4, if all the heat produced in the bearings is carried away by a stream of water flowing at 18 gallons per hour, find the temperature of the outflowing water, the supply being at 50° F.

[1 B.TH.U. = 778 ft.-lbs.; 1 gallon of water weighs 10 lbs.]

Ex. 6. Why is a bearing for a shaft always designed for as small a diameter as is consistent with adequate strength?

Ex. 7. In a test of a small engine 70 lbs. were loaded on one side of the flywheel brake, and the other side was attached to a spring balance reading 11 lbs. The speed of the engine was 200 revs. per min., and the brake wheel was 5 ft. diameter. Calculate the heat generated at the brake in 10 mins.

Ex. 8. A truck of mass 3 tons is running free down a slight incline with a uniform speed of 15 miles an hour, when the brake is put on. The brake consists of a wooden block pressed on the iron rim of one wheel. Find what pressure must be applied to the block to bring the truck to rest in 40 yds., the coefficient of friction for the wood on iron being ·3, and less than the coefficient of friction between the wheels and rails.

Ex. 9. A ship of 2000 tons moving at 3 knots is stopped in 152 ft. Find the retarding force, assuming it to be constant.

[1 knot = 6080 ft. per hour.]

Ex. 10. The ram of a pile driver weighs 200 tons and falls 12 ft. on the head of a pile, which yields half an inch. Find the loss of potential energy of the ram, and assuming that it all goes in overcoming the resistance of the pile, find the steady weight which the latter could support.

Ex. 11. A battleship is steaming at the rate of 25 miles an hour and her engines are developing 30,000 horse power at the propeller. What is the resistance to her motion through the water? In what distance would she lose all her way if retarded with this force, assuming that her mass is 15,000 tons?

Ex. 12. A chain will be stretched across the locks in the Panama Canal, 70 ft. from the end of the lock; this chain opposes a resistance to a ship which strikes it, and so brings it to rest before reaching the end gate. If this brake is sufficient for a 10,000 ton ship reaching the chain at a speed of 4 miles an hour, at what speed may a 30,000 ton ship safely reach the chain? (The friction of the water may be neglected at these speeds.)

Ex. 13. A collier is being towed by means of a hawser, the tension in the hawser being 10 tons and the speed 3·5 knots (1 knot = 6080 ft. per hour). What H.P. is being spent in towing the collier?

Ex. 14. In measuring Joule's equivalent, the number of revolutions was 2725; the rise of temperature 5·33° C.; the water had an effective mass of 270 grms.; diameter of drum 24 cm.; difference between the tensions on the two sides of the brake was 300 grms. weight. Find these quantities in British units.

Find the B.TH.U.'s developed, the work expended in ft.-lbs., and the equivalent of 1 B.TH.U. in ft.-lbs.

Ex. 15. A gun is fired at an elevation of 30°, and is found to have a range of 11,500 yds. Neglecting the resistance of the air, calculate the muzzle velocity. If the mass of the projectile is 700 lbs., find its kinetic energy when at its highest point.

Ex. 16. A swing is pulled from its lowest position till the ropes, which are 10 ft. long, make an angle of 40° with the vertical, and then let go. Calculate its speed as it passes through its lowest point. If the boy on it weighs 120 lbs., what constant force must be exerted on him to stop him in 2 ft. after passing this point (neglecting the rise in this distance).

Ex. 17. A bicyclist propels his machine at 10 miles an hour against a road resistance of 4 lbs. wt. His pedal-crank is 7 ins. long, the number of teeth on the crank axle chain wheel is 45 and on the hub chain wheel is 18, and the diameter of the back wheel is 28 inches. Calculate (i) the distance moved by the bicycle for one revolution of the crank axle, (ii) the work done against the road resistance in this distance, and hence (iii) the average force perpendicular to the crank which the rider must exert.

Ex. 18. A flywheel has 300,000 ft. lbs. of K.E. stored in it when rotating at 100 revs. per min.; how much work will be done by it in slowing down to 98 revs. per min.?

Ex. 19. A flywheel whose mass is 60 lbs. is mounted on a horizontal axle of diameter $1\frac{1}{4}$ ins., round which a string is coiled and attached to a weight of 4 lbs. If the weight falls 30 ins. in 10 secs. from rest, find the radius of gyration of the flywheel.

Ex. 20. A hollow drum, 6 ft. in diameter, and mass 5 cwt., has a radius of gyration of 2 ft. It is employed to wind a load of 500 lbs. up a vertical shaft, and is rotating at 120 revs. per min. How far below the ground level should the load be when the steam is shut off, so that the kinetic energy of load and drum should just suffice to carry the former to the surface?

Ex. 21. A flywheel weighing 5 tons has a radius of gyration of 5 ft.; it is carried on a shaft of 12 ins. diameter and is running at 75 revs. per min. If the coefficient of friction of the shaft in its bearings is ·07, how many revolutions will the flywheel make before it is stopped by friction?

Ex. 22. A lift of mass 300 lbs. is started from rest by a constant force and acquires an upward speed of 20 ft. per sec. in 3 secs. What is then its kinetic energy? What is the total work done on the lift in these 3 secs.?

Ex. 23. A car on a switch-back railway is started from a height of 40 ft. above the ground. It goes down into a dip and reaches the top of the next rise, 30 ft. above the ground, with a speed of 8 ft. per sec.; the distance along the rails between the crest of this rise and the starting point is 38 yds. Calculate the force of friction (supposed constant) which opposes the motion of the car, the mass of the car and passengers being 2000 lbs.

If the total length of the rails is 130 yds., and the car is required to come to rest at the end, what must be the height above the ground of the finishing point?

Ex. 24. A flywheel, whose diameter is 7 ft., mass 4 tons and radius of gyration 3 ft., is making 90 revs. per min. Calculate how many ft.-lbs. of work it will do in coming to rest. What force acting along its circumference would be required to stop the wheel in 10 revolutions?

Ex. 25. A flywheel has a mass of 5 cwt. and gives up 1800 ft.-lbs. of energy while its speed drops from 140 to 133 revs. per min. What is its radius of gyration?

Ex. 26. A body of mass 20 lbs. is projected with a velocity of 80 ft. per sec. at an elevation of 50°. Calculate the potential and kinetic energy of the body after a second.

Ex. 27. Two wheels, each of mass 30 lbs. and each mounted on a 3 inch axle, have 10 lb. weights attached to cords round the axles at heights of 10 ft. above the ground. The weight attached to one reaches the ground in 20 secs. from rest, that attached to the other in 40 secs. from rest. Compare the radii of gyration of the two wheels, neglecting the K. E. of the weight in each case.

Ex. 28. A railway coach, mass 5 tons, is slipped from a train travelling at 30 miles an hour and comes to rest in 400 yds. Determine the original kinetic energy of the coach, and the force (assumed constant) which brings it to rest.

Ex. 29. A flywheel has a mass of 4 cwt. and a radius of gyration of 3 ft. Calculate the energy stored in it when making 240 revs. per min. What will be its speed when it has given up 25 % of this energy?

Ex. 30. Water enters a pipe-line whose area of cross-section is 10 sq. ft. at a speed of 70 ft. per min. It flows down to a turbine 50 ft. below, and emerges from it at a speed of 20 ft. per min. Calculate the loss of energy, kinetic and potential, per lb. of water. If 80 % of this is utilised by the turbine, calculate its horse-power. (1 cub. ft. of water weighs 62·4 lbs.)

Ex. 31. A man works a treadle lathe and drives the flywheel at a speed which fluctuates because the man does work with his foot during one half only of each revolution of the flywheel; the average rate at which he works is one-tenth of a horse-power. Assuming that the lathe absorbs energy at a uniform rate throughout, and that the man supplies it at a uniform rate during one half of each revolution and none during the other half, calculate the necessary radius of gyration of the flywheel (whose mass is 160 lbs.) in order that its speed may vary between 105 and 95 revs. per min. The effect of parts of the lathe other than the flywheel in keeping the speed uniform may be neglected.

Ex. 32. The travelling table of a planing machine has a mass of 35 cwts., and during the cutting stroke it moves at a speed of 21 ft. per min.; it is driven by gearing from a shaft carrying two pulleys, of diameter 12 ins. and 21 ins. and masses 6 lbs. and 12 lbs. respectively; these masses may be considered as being concentrated on the rims. During the cutting stroke this shaft turns at 248 revs. per min. Calculate the kinetic energy stored in (1) the table, (2) the pulleys. If the coefficient of friction between the table and the bed on which it slides is ·1, calculate the distance which the table will run after the driving belt has been thrown off the pulley, and the time occupied in stopping.

Ex. 33. In the planing machine of Ex. 32, 26 % of the power exerted by the driving motor reaches the table; calculate the H.P. of the motor needed to drive the table against frictional resistance when no cut is being made. When a cut is being made in cast iron $\frac{1}{8}$ in. deep, it is observed that the H.P. exerted by the motor is 2·04; calculate the resistance opposed to the cutting tool, if the object being planed weighs 5 cwt.

CHAPTER VII

MOMENTUM

77. Action and Reaction; Newton's Third Law of Motion. Hitherto we have considered the effect of a force on a body, without troubling ourselves about the manner in which the force is produced. But that force must be exerted by some other body, and we will now consider how this body is affected. One body can exert a force on another by pushing directly against it, or by pulling it by a rope, or by means of "action at a distance" such as the attraction of gravitation or electric or magnetic attraction. In all these cases the body which exerts the force is itself affected by it; for example if a heavy body lies on the table, it presses the table down and the table presses the body up and so supports it.

When the two bodies are at rest it may be taken as obvious that the "action" of the first on the second is equal to the "reaction" of the second on the first; this is assumed in every problem in Statics. It may not be so obvious when the bodies are moving under the action of the forces they exert on one another.

When dealing with the action between two bodies which are changing their speeds, it is convenient to have a definite case to think about. The following experiment will serve this purpose.

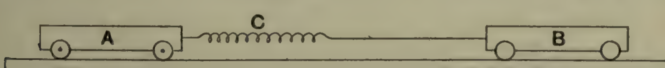


Fig. 55.

A and *B* (Fig. 55) are two trolleys on a plane. To *A* is fixed one end of a spiral spring, *C*, the other end of which is attached to *B* by a long thread. The trolleys are held apart so that the spring is stretched; then *B* is being pulled to the left by the spring, and at the same time *A* is being pulled to the right, and these forces are obviously equal since *A* and *B* and the spring are at rest. If we look on the spring as being part of *A*, we may regard the two bodies as exerting a force on each other. They are simultaneously let go; while the spring is recovering its original unstretched length, *A* and *B* move towards one another, with increasing speeds; so each must continue to exert some force on the other during the motion, until the spring has recovered its original length.

Newton asserted that *A* exerts on *B* a force equal at every instant to that which *B* is then exerting on *A*. He put it more generally in his Third Law of Motion. **"To every action there is an equal and opposite reaction, or, the mutual actions between any two bodies are always equal and opposite."** This need not be proved experimentally in the same way as the Second Law; it is a generalisation of our everyday experiences, and is a consequence of the meaning which we give to the words force, action and reaction. A few instances may make it clearer. A rifle bullet can exert no force until it meets with an obstacle to its further progress; you cannot drive a nail into a board unless it is supported so that it cannot yield, for the hammer cannot exert a pressure unless the board opposes the intrusion of the nail. A multitude of instances such as these make it clear that "to every action there is an opposite reaction," but it is not so obvious that the reaction is always equal to the action. In fact, it may look at first sight as though, if this were so, no body would ever succeed in making another move at all. Newton put this difficulty clearly by quoting the case of a cart and horse; according to his third law, the cart pulls the horse backwards as hard as the horse pulls the cart forwards, and yet the horse succeeds in making the cart move forward, and the cart does not succeed in making the horse move backward.

78. To clear up this apparent paradox we have only to do what is very advisable in all problems of dynamics; first be clear as to what body or collection of bodies we are thinking about and then reckon up *all* the forces which are acting on that body or collection of bodies, from outside. Applying the Third Law in the case of each of these forces, we can show that both cart and horse may move forwards.

Consider first the **cart**, and suppose that it is on a level road. The forces on it from outside are, if it is moving forwards,

(1) the vertically downward attraction of the earth on it,

(2) the vertically upward pressure of the ground on the wheels,

(3) the horizontal pull of the traces, forwards,

(4) certain frictional resistances, acting horizontally backwards, by the ground on the wheels. For simplicity we can think of these as equivalent to a direct pull backwards by a rope.

Forces (1) and (2) must be equal, or the cart would rise into the air or sink into the ground with an acceleration. Forces (3) and (4) need not be equal; they are not equal if the cart has an acceleration, for by the second law of motion the difference of these forces is equal to the product of the mass of the cart and its acceleration, and as we are assuming that the cart has a forward acceleration, (3) is greater than (4).

Next consider the forces acting from outside on the **horse**; they are

(5) the vertically downward attraction of the earth on him,

(6) the vertically upward pressure of the ground on his hooves,

(7) the horizontal pull of the traces, backwards,

(8) certain frictional forces, acting horizontally forwards, by the ground on his hooves.

Forces (5) and (6) must as before be equal to one another. Forces (7) and (8) are not equal to one another; their difference is equal to the product of the mass of the horse and his acceleration, and as we assume that the horse has a forward acceleration (8) must be greater than (7). These frictional forces exerted on the horse by the ground are the reaction to the backward force which he exerts on the ground in his efforts to move forward; he succeeds in moving forward because the force which he exerts is greater than the backward pull of the traces.

According to the Third Law, (3) and (7) are equal to one another; and it will be seen that this fact does not interfere with our explanation of the movement of either the cart or the horse, considered separately.

Lastly, consider the **horse and cart** as one "system of bodies." The forces acting on it from outside are,

(9) the vertically downward attraction of the earth on the two,

(10) the vertically upward pressures of the ground on wheels and hooves,

(11) the backward force of friction by the ground on the wheels,

(12) the forward force of friction by the ground on the horse's hooves.

As before (9) and (10) balance one another. Since the whole system has a forward acceleration, (12) is greater than (11). (This is what we found when we considered the horse and cart separately, for we found that (8) was greater than (7), and (3) than (4), and the third law states that (7) and (3) are equal; hence (8) is greater than (4).) It will be noticed that in this case the pull of the traces does not come into question; it is no longer a force from outside, but a reaction between two parts of the system, which cannot affect the motion of the system as a whole any more than a passenger in the cart could do by pulling the dashboard towards him.

So we see that the Third Law does not really lead us into the absurdity which at first sight it seemed to do.

79. Atwood's Machine. We will now work out numerically a simple instance of the application of the Third Law of Motion, in which two bodies move under the attraction of the earth and their reaction on one another.

Suppose that two unequal weights A and B are hung at the ends of a light string passing over a light pulley, which moves with so little friction in its bearings that we can neglect it. This arrangement is usually called "Atwood's Machine." Suppose the mass of A to be 400 grms., and of B to be 410 grms.; it is obvious that B will run down, and that A will run up. Since the string is supposed not to stretch, the speed of A and B at any instant must be the same, so they have the same acceleration; call the acceleration a cm. per sec. per sec.

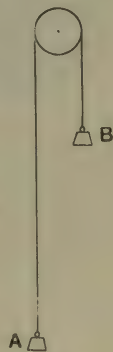


Fig. 56.

A is being pulled upwards by the string, with a force whose magnitude we do not yet know; call it T dynes. It is also pulled downwards by the earth with a force of 400×981 dynes; these forces are shown in Fig. 57 (i).

We know that A is moving upwards with an acceleration a cm. per sec. per sec.; the resultant force on it to produce this acceleration must be $400 \times a$ dynes, as shown in Fig. 57 (ii).

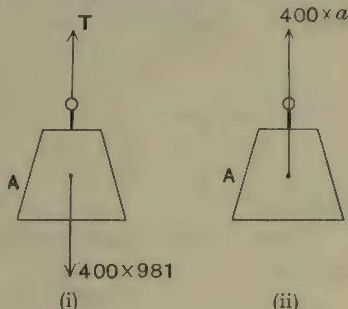


Fig. 57.

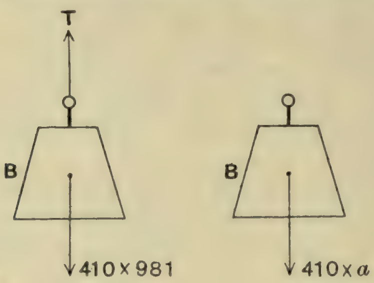
It will be found to be advisable, in all problems of this kind, to isolate in some such way the body under consideration, and draw separate diagrams for it showing "applied forces" and "accelerating force," in order to avoid leaving anything out of account.

We see from Fig. 57 (i) that the upward resultant force on A is $T - 400 \times 981$ dynes; hence

$$T - 400 \times 981 = 400 \times a \quad \text{.................(1).}$$

We have called the pull of the string on A, T dynes; this force was of course exerted by B and transmitted by the string; hence by Newton's Third Law, the reaction of A on B, transmitted to B by the string, is also T dynes. So the string pulls B upwards with a force of T dynes.

Hence Fig. 58 (i) represents the diagram showing forces on B;



(i) (ii)

Fig. 58.

and Fig. 58 (ii) represents its diagram of acceleration. From these as before we see that

$$410 \times 981 - T = 410 \times a \quad \text{.................(2).}$$

Adding the respective right and left sides of (1) and (2) together we get

$$410 \times 981 - 400 \times 981 = 410a + 400a$$

or $$10 \times 981 = 810a,$$

whence $$a = \mathbf{12 \cdot 1} \text{ cm. per sec. per sec.}$$

Again, substituting this value of a in (1), in order to find T, we get

$$T - 400 \times 981 = 400 \times 12{\cdot}1,$$

or $$T = 400 \times 993{\cdot}1,$$

$$= \mathbf{397{,}200} \text{ dynes.}$$

It will be seen that the tension of the string is intermediate between the weights of A and B; for the weight of A is 400×981 or **392,400**, and of B is 410×981 or **402,300** dynes. The resultant force on A is $397{,}300 - 392{,}400$ or **4,900** dynes upwards, and the resultant force on B is $402{,}300 - 397{,}300$ or **5,000** dynes downwards; these respective forces give A and B the same acceleration, 12·1 cm. per sec. per sec.

Ex. 1. Masses of 50 and 60 grms. are hung by a light string over a light pulley; determine the resulting acceleration and the tension of the string.

Ex. 2. Masses of 20 and 21 lbs. are hung over a pulley whose mass and friction may be neglected; determine the acceleration, and the tension of the string in poundals.

80. It is convenient to get a general value of the acceleration and tension in the string, for any values of the weights A and B. Suppose the mass of A is m_1 grms. (or lbs.) and of B is m_2 grms. (or lbs.); call the acceleration a cm. per sec. per sec. (or ft. per sec. per sec.) and the pull in the string T dynes (or poundals). We must of course use either British or C.G.S. units throughout. Working as before, we have

$$T - m_1 g = m_1 a \quad \text{.......................(1)},$$

and

$$m_2 g - T = m_2 a \quad \text{.......................(2)},$$

where g as usual means the acceleration of a body falling freely, expressed in the same units as we are using for T, etc.

Hence as before

$$a = \frac{m_2 - m_1}{m_2 + m_1} \times g \quad \text{.......................(3)},$$

and

$$T = \frac{2m_1 m_2}{m_1 + m_2} g \quad \text{.......................(4)}.$$

These results (cf. Ex. 15 of Chap. VI) should not be learnt as formulae; the same method of working should be followed with the numbers given in any particular case.

Writing (3) in the form

$$(m_2 + m_1)\, a = (m_2 - m_1)\, g$$

we see that the acceleration a is the same as would be produced by a force $(m_2 - m_1)\, g$ acting on a mass of $(m_2 + m_1)$; this force is the difference of the weights, and this mass is the total mass moved; so the acceleration can be determined at once by considering the system as a whole, and taking the acting force as the difference of the weights and the total mass as the body on which this force acts. This way of looking at it is obviously reasonable and convenient for future use but it would not be safe to assume that it would give the true result without proving, as we have done, that it does so.

A form of Atwood's machine which is convenient for experimental purposes consists of two masses connected by a ribbon of paper, which passes under a vibrating spring, as in a trolley. The whole motion is then recorded on the paper; the value of the acceleration above deduced should be verified if such a machine is available.

Ex. 3. Weights of 20 and 21 lbs. are hung by a light string over a light frictionless pulley; calculate the acceleration and the tension of the string, using the general principle found in the last article.

Ex. 4. A lift weighing 8 cwt. is counterpoised by an equal weight, and a man weighing 10 stone steps into it; if it runs down freely determine its acceleration.

Ex. 5. Calculate the speed of the lift in Ex. 4 when it has fallen 20 ft.

Ex. 6. Weights of 600 and 650 grms. are hung by a light string over a light frictionless pulley; calculate the acceleration and the tension of the string, using C.G.S. units.

81. Tension of the string by which a weight pulls a trolley. As a further example we will investigate the results discovered by experiment in Art. 13. Suppose the mass of the trolley is 6 lbs., and that we have counteracted friction by sloping the plane; suppose that the mass of the acting weight is ·5 lb. If we call the tension of the string T poundals, and the acceleration a ft. per sec. per sec., the diagrams showing force and acceleration for the weight are as shown in Fig. 59.

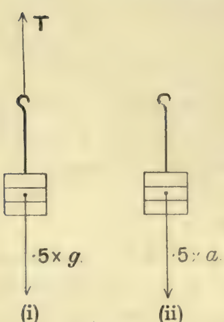


Fig. 59.

Hence $\cdot 5g - T = \cdot 5 \times a$ ....(1).

The corresponding diagrams for the trolley are given in Fig. 60, where the slope of the plane is exaggerated in order

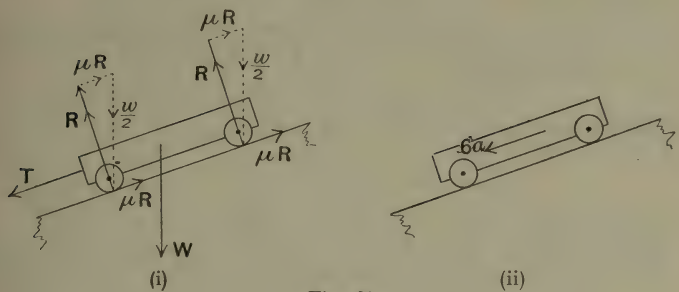


Fig. 60.

to show its effect. Here the weight of the trolley balances the forces (R and μR) exerted by the plane at each of the wheels (as shown by dotted lines), leaving as the resultant force on the trolley the pull of the string, i.e. T poundals down the plane.

Hence $T = 6a$ ....(2).

Eliminating a from (1) and (2) we have

$$\frac{T}{6} = a = \frac{\cdot 5g - T}{\cdot 5} = g - \frac{T}{\cdot 5},$$

whence $T = \frac{6}{13} \times g = \mathbf{\cdot 461g}$ poundals.

So the pull on the trolley is less than the weight of the acting mass (which is $\cdot 5g$ poundals) by $\cdot 039$ poundals.

If we work out the pull on a trolley of mass 3 lbs. when the acting mass is ·5 lb. as before, we find that the pull is $\frac{3}{7} \times g$ or **·428g** poundals, i.e. the pull decreases as the mass of the trolley decreases, as we found in Art. 13. Note that if the mass of the trolley is reduced to the vanishing point, there will be no pull in the string, for it then has nothing to pull.

Ex. 7. A trolley stands on a plane whose gradient is 1 in 4; the mass of the trolley is 2000 grms. A mass of 700 grms. hangs on the end of a string which passes over a pulley at the top of the plane and is attached to the trolley. Neglecting friction, find the acceleration produced.

Call the tension of the string T dynes, and the acceleration a cm. per sec. per sec.

The forces on the trolley parallel to the plane are the resolved part of the weight of the trolley down the plane, or $\frac{1}{4} \times 2000 \times 981$ dynes, and T dynes upwards; so we have

$$T - \tfrac{1}{4} \times 2000 \times 981 = 2000 \times a \text{ ....(1).}$$

The forces on the mass are its weight downwards, or 700×981 dynes, and T dynes upwards, so

$$700 \times 981 - T = 700 \times a \text{ ....(2).}$$

From (1) and (2) we find the acceleration is **72·7** cm. per sec. per sec.

Ex. 8. A trolley of mass 4 lbs. stands on a plane whose gradient is 1 in 3; a weight of 1·5 lb. hangs on the end of a string which passes over a pulley on the top of the plane and is attached to the trolley. Neglecting friction, find the acceleration produced.

Ex. 9. Work out Ex. 8 if the acting weight is 1·0 lb. instead of 1·5 lb.

***82.** As a further example of Newton's Third Law, take a more complicated set of masses and pulleys, as shown in Fig. 61. The axle of the upper pulley is supposed to be fixed, the lower being hung on the rope carrying A. The masses of all ropes and pulleys, as well as friction, may for simplicity be disregarded.

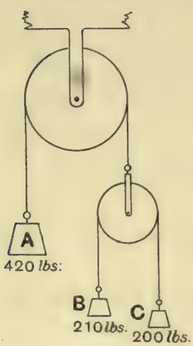


Fig. 61.

Call the tension in the upper rope T_1 poundals, and draw the force and acceleration diagrams for A as in Fig. 62 (i) and (ii), i.e. we assume A has a downward acceleration of a_1 ft. per sec. per sec. Then we can write down the "equation of motion" of A as

$$420 \times 32{\cdot}2 - T_1 = 420 \times a_1 \text{ ....(1).}$$

Fig. 62 (iii) represents the force diagram for the lower pulley; as its mass is negligibly small, there can be no resultant force on it, so

$$T_1 = 2T_2 \text{ ....(2).}$$

Here we have denoted the tension of the lower rope by T_2 poundals; it must be equal on both sides of the pulley as there is nothing to cause a change of tension in passing over the pulley. And by Newton's Third Law the upward pull of the upper rope on this pulley must be the same as on A, i.e. T_1 poundals.

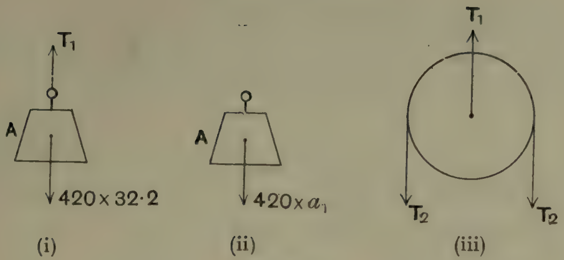


Fig. 62.

Call the upward acceleration of B and C a_2 and a_3 ft. per sec. per sec. respectively; then their force and acceleration diagrams are as in Fig. 63 (i) and (ii).

So their equations of motion are

$$T_2 - 210 \times 32{\cdot}2 = 210a_2 \quad \text{.........................(3)},$$

$$T_2 - 200 \times 32{\cdot}2 = 200a_3 \quad \text{.........................(4)}.$$

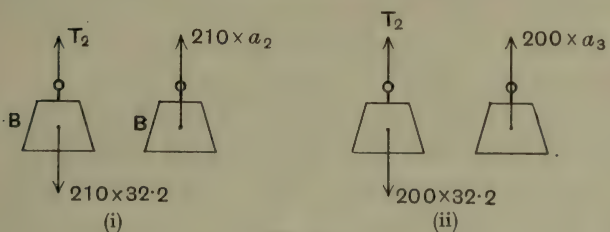


Fig. 63.

There must be some connection between the accelerations a_1, a_2 and a_3; we shall most easily discover it imagining the bodies to start from rest and finding how far each will have moved in 1 second. A's average speed downward during this time is $\frac{a_1}{2}$ ft. per sec., so it will have dropped $\frac{a_1}{2}$ ft.; hence the lower pulley will have risen $\frac{a_1}{2}$ ft. Similarly B will have risen $\frac{a_2}{2}$ ft., so the length of rope between B and the lower pulley will have lengthened by $\left(\frac{a_1}{2} - \frac{a_2}{2}\right)$ ft. Similarly the length of rope between C and the lower pulley will have lengthened by $\left(\frac{a_1}{2} - \frac{a_3}{2}\right)$ ft. Now the length of this rope does not change, so

$$\left(\frac{a_1}{2} - \frac{a_2}{2}\right) + \left(\frac{a_1}{2} - \frac{a_3}{2}\right) = 0,$$

or

$$2a_1 = a_2 + a_3 \quad \text{.................................(5).}$$

From these five equations we can calculate each of the three accelerations and the tensions of the two ropes.

***Ex. 10.** Find the acceleration of A in above case.

83. Momentum. Repeat the experiment described in Art. 77, but remove the effects of friction by putting each trolley on a separate plane, sloped so that they run down with

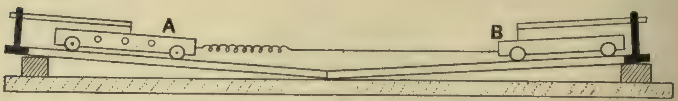


Fig. 64.

constant speed. Hold the trolleys apart, and let them go simultaneously; take the tracings and determine from them the value of the velocity of each after the spring between them has recovered its original length; these velocities will then be constant.

In a particular case, the mass of A was 2370 grms., the mass of B was 3490 grms.; the period of the spring was $\frac{1}{5}$ sec.; the final speed of A was found to be 6·80 cm. in $\frac{1}{5}$ sec., or **34** cm. per sec.; the final speed of B was found to be 4·60 cm. in $\frac{1}{5}$ sec. or **23** cm. per sec.

We know that the forces acting on A and B were at every instant the same, and of course A and B were under the action of these forces for the same length of time, so the effect on A and B should be the same. A and B have not acquired the same speed; have they acquired the same kinetic energy? The final kinetic energy of A is $\frac{1}{2} \times 2370 \times 34^2$ ergs, or **1,370,000** ergs; the final kinetic energy of B is $\frac{1}{2} \times 3490 \times 23^2$ ergs, or **923,000** ergs. The difference between these results is clearly greater than any probable error of experiment, so we must admit that under these circumstances the two bodies do not acquire the same energy.

But try another possible way of representing the "quantity of motion" of a body; multiply together the numbers representing its mass and its velocity. Doing this for A we get as a result 2370×34 or **80,580**, and for B we get 3490×23 or **80,260**. These differ by less than $\frac{1}{2}$ per cent., which is of course well within the error of experiment. So it appears that under some circumstances this is a suitable way to represent the quantity of motion in a body.

A special name "**Momentum**" is given to the product of the number of units of mass into the number of units of velocity of a body.

We know (Arts. 42 and 65) that if the same (or equal) forces act on two bodies *through the same displacement*, the two bodies acquire the same increase of *energy*, whatever their masses; we have just seen a case in which the same forces act on two bodies of different masses *for the same time*, and the two bodies acquire the same increase of *momentum*.

The momentum of a body is one way of measuring the "quantity of motion" it has in it, the kinetic energy being another way; it is better to use sometimes one, sometimes the other, according to the problem to be solved. Care has always to be taken in deciding which to select; no general rule can be laid down to cover all cases, though momentum is usually the better when the *time* of action of the force is given, energy when the *displacement* under the action of the force is known and we are sure that no energy is lost by friction, etc. No name is given to the unit momentum; but the units in which the momentum is expressed must be stated, usually "foot-pound-second" or "C.G.S." units are used. Then a momentum of "50 f.p.s. units" would mean the momentum of a mass of

50 lbs. moving at 1 ft. per sec., or of a mass of 25 lbs. moving at 2 ft. per sec., and so on.

Ex. 11. What is the momentum of a mass of 5 lbs. moving with a speed of 20 ft. per sec.?

Ex. 12. What is the momentum of a mass of 200 tons moving at a speed of 60 miles an hour? (Express in f.p.s. units.)

Ex. 13. A body of mass 80 tons has a momentum of 254,285 f.p.s. units; what is its speed?

Ex. 14. A 12 stone man runs at 10 miles an hour; what is his momentum? With what speed must an 8 stone man run, in order to have the same momentum?

Ex. 15. Will the two men in Ex. 14 have the same kinetic energy?

84. Momentum produced by a constant force. If we know that the force on a body is constant, and we also know the length of time during which the force acts, we can easily calculate the momentum produced. Consider for example a shell, weighing 1 cwt., which is being fired from a gun; suppose that the powder burns slowly enough to keep up a constant pressure on the base of the shell while it is within the gun, and that this pressure is 100 tons wt. (or $100 \times 2240 \times 32{\cdot}2$ poundals); and suppose the shell takes $\frac{1}{28}$ sec. to travel up the gun. We know that the force equals the product of the numbers representing the mass and acceleration; so the acceleration is $\frac{100 \times 2240 \times 32{\cdot}2}{112}$ ft. per sec. per sec., or 64,400 ft. per sec. per sec. Hence the speed attained at the end of $\frac{1}{28}$ sec. from rest is $\frac{1}{28} \times 64{,}400$, or 2300, ft. per sec., and the momentum attained by the shell is 112×2300 f. p. s. units, or **257,600** f. p. s. units.

By Newton's Third Law, the force on the gun is also 100 tons wt., and it clearly acts for the same time, $\frac{1}{28}$ sec.; hence if the gun is free to recoil and if we know its mass (80 tons, say), we can in the same way calculate its final speed and momentum.

These are $\frac{1}{28} \times \frac{100 \times 2240 \times 32 \cdot 2}{80 \times 2240}$, or 1·4375, ft. per sec. for the speed, and $80 \times 2240 \times 1 \cdot 4375$ or **257,600** f. p. s. units for the momentum. As before, we find that the momentum of each of the two bodies which reacted on one another is the same, but in opposite directions.

We can give a simple general proof of the truth of what has just been laboriously calculated out in a particular case, as follows.

85. If we denote the mass of a body by m, the constant force on it by F, and the acceleration produced in it by a, using either British or C.G.S. units, we know (Art. 27) that these numbers are connected by the equation $F = ma$. Now the acceleration means the increase (or decrease) of velocity each second; so ma measures the increase (or decrease) of the momentum of the body each second. Hence if the constant force F acts on the body for one second, the increase (or decrease) of momentum is ma, that is F; if it acts for t secs., the change of momentum is $ma \times t$, or Ft. So if a constant force F acts on a body for t secs., the change of momentum produced is measured by Ft; or in words, *the change of momentum in a body under the action of a constant force is measured by the product of the numbers representing the force and the time during which it acts.*

Ex. 16. A mass of 2 kilograms falls freely from rest for $\frac{1}{3}$ sec.; what momentum does it acquire?

Its acceleration is 981 cm. per sec. per sec., so its velocity after one third of a second is 327 cm. per sec. Hence its momentum is 654,000 C.G.S. units.

Ex. 17. A train weighing 200 tons is moving at 60 miles an hour; what is its momentum? It is brought to rest by a constant braking force in one minute; what is the braking force?

Its momentum is $200 \times 2240 \times 88$ f.p.s. units. If F poundals is the braking force, the product of the force and the time during which it acts is $F \times 60$ in British units, so $60F = 200 \times 2240 \times 88$; hence $F = 657{,}000$ poundals or 20,400 lbs. wt.

Ex. 18. A motor car weighing 1 ton is moving at 20 miles an hour, and is brought to rest by a constant braking force in 10 secs.; what is the braking force?

Ex. 19. If the greatest braking force that can be exerted on the train in Ex. 17 is one seventh of its dead weight, what is the shortest time in which it can be brought to rest?

Ex. 20. How far will the train in Ex. 19 run before stopping?

When two bodies act on one another with a constant force, we know that the action and reaction are equal and opposite, and these forces must be in action on both bodies for the same time; hence from the last paragraph, if no other resultant forces act on them, *the change of momentum in each must be the same.*

We may put this in another way; when two bodies act on one another with a constant force for any length of time the loss of momentum in one body is equal to the gain of momentum in the other. Here we are regarding the momentum as being positive in one direction and negative in the other, like velocity.

Ex. 21. A rifle of mass 8 lbs. fires a bullet of mass 1 oz. with a muzzle velocity of 2400 ft. per sec. Find the speed of recoil of the rifle, if unresisted. Find also the kinetic energy of the rifle and bullet.

The momentum of the bullet is $\frac{1}{16} \times 2400$ or 150 f.p.s. units; this must therefore be the backward momentum of the rifle; so its velocity is $\frac{150}{8}$ or **18·75** ft. per sec.

The kinetic energy of the bullet is $\frac{1}{2} \times \frac{1}{16} \times 2400^2$ or **180,000** ft.-poundals; the kinetic energy of the rifle is $\frac{1}{2} \times 8 \times 18{\cdot}75^2$ or about **1406** ft.-poundals. So the bullet carries off nearly all the **energy** of the explosion although the **momentum** of the bullet and rifle are the same.

Ex. 22. A gun weighing 6 tons fires a 12 lb. shot with a muzzle velocity of 1800 ft. per sec.; find the speed of recoil of the gun if unresisted.

Ex. 23. Compare the momentum of a 12 stone man running 100 yds. in 10 secs. with the momentum of a rifle bullet of mass 1 oz. moving at 2400 ft. per sec. Compare also their kinetic energies.

Ex. 24. A 10 stone man standing on smooth ice discharges in a horizontal direction a rifle weighing 7 lbs. The bullet weighs 1 oz. and has a muzzle velocity of 2200 ft. per sec. With what velocity will the man recoil?

86. Sign of Momentum. It is important to note that momentum, like velocity but unlike kinetic energy, has *direction* as well as magnitude; in particular, if we are considering motion in one straight line only, a body moving in the direction which we take as the positive direction has positive momentum, and if it moves in the opposite direction its momentum is negative. So if a rifle containing a cartridge is at rest, and is fired, the final momenta of the rifle and cartridge are together zero, since they are equal in magnitude and opposite in sign.

The velocity of a body can be resolved into two components, so the momentum can also be resolved into two components. If a force acts on a body already in motion, it produces a corresponding change of velocity *in the direction of the force* (see Art. 41), and therefore it changes the component of the momentum in that direction only.

87. Collision. It is a common occurrence for a body in motion to act on another body at rest, and the force between them does not usually remain constant; for example, a billiard ball striking another, a foot striking a football, a hammer striking a nail, wind blowing on a sail, etc. In all such cases the loss of momentum by one body is equal to the gain of momentum by the other, even if the force between them varies during the time in which they react on one another. If we knew the magnitude of this force at every instant during the reaction, we might be able to calculate the change of momentum it would produce; but in the majority of such cases we cannot tell accurately how the force varied, nor what its magnitude was at each instant, nor even how long the bodies were acting on one another. But theory and experiment both show that the final result of the action is that one body gains exactly the same amount of momentum as the other loses, and we are chiefly concerned with the final result. We will now show this by some experiments.

Experiment i. Take two trolleys, A and B (Fig. 65), on a plane sloped to balance frictional resistances. They are connected by a thread and spiral spring, as described in Art. 77. *B* is initially at rest, and *A* moving at uniform speed to the right; the thread is at first slack. When the thread becomes taut and

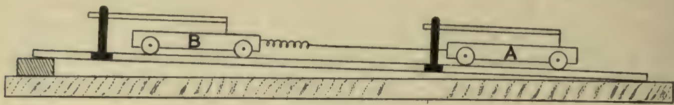


Fig. 65.

the spring begins to stretch, *B* begins to experience a force urging it to the right, and *A* a retarding force which at every moment is equal to the accelerating force on *B* at the same moment. This force increases as the spring is stretched further, until it reaches a maximum when *B* is moving at the same speed as *A*; after this, *B*'s speed becomes greater than *A*'s, and the extension of the spring decreases as *B* closes up on *A*, until finally the string becomes slack again, and *B* and *A* move each with its own constant velocity; *B*'s speed being the greater. *B* will of course subsequently catch up *A* and collide with it, but we will not for the present concern ourselves with that.

The following are the results of an experiment to illustrate this. *A* weighed 7 lbs. 9 oz., and *B* weighed 5 lbs. 3 oz., and *A*'s initial speed was observed to be ·4825 ft. per sec. After the spring had ceased to act, *A*'s final speed was found to be ·0910 ft. per sec., and *B*'s was ·572 ft. per sec., all in the same direction. Hence *A*'s initial momentum was $7\frac{9}{16} \times \cdot 4825$ or 3·65 f.p.s. units, and its final momentum was $7\frac{9}{16} \times \cdot 0910$ or ·688 f.p.s. units; so it lost **2·96** f.p.s. units. *B* gained $5\frac{3}{16} \times \cdot 572$ or **2·97** f.p.s. units. The difference between the gain and loss is within the error of observation.

Another experiment illustrates it; *A* and *B* were adjusted to have equal masses, and *A* was brought to rest, and *B* took up *A*'s original speed; in this case it is obvious that *B*'s gain of momentum equals *A*'s loss.

If B is a good deal heavier than A, then A's final velocity will be in the opposite direction; but it is not easy to arrange an experiment to verify the principle numerically in this case, as the slope of the plane necessary to counteract friction is against A when its velocity is reversed, so this velocity is not constant and cannot be measured. But A will be seen to start up the plane as the spring relaxes, though it soon comes to rest.

88. We can see that these experimental results must be true, from theoretical considerations, as follows. Although while the spring is stretched the force it exerts is continually changing, yet at any moment it is exerting the same forces on A and B; so it may be considered to be obvious that during any period while the force is in action the increase of A's momentum is equal to the (numerical) increase of B's momentum, since we have shown that, at any rate for a constant force, this change of momentum is equal to the product of the force and the time of action.

If further explanation of this be needed, consider any short period of the motion, say one millionth of a second; suppose that the spring is extended altogether during two seconds, exerting a force of six poundals when at its maximum. Suppose at the beginning of the millionth of a second the force is 4 poundals, and at the end it is 4·00018 poundals; since the force never differs from 4 poundals by more than ·00018 poundal, we shall not make an error of 2 in 40,000, or one two hundredth per cent., if we consider the force during the millionth of a second to be constant (and equal to 4 poundals), and hence take the increase in B's momentum, and decrease in A's momentum, to be equal to one another (both being $4 \times \frac{1}{1,000,000}$ f.p.s. units) during this period. If we take a shorter period, the percentage error will be still smaller. So we can cut up the time into very small intervals in each of which B gains as much momentum as A loses, to any required degree of accuracy; adding these changes together for the whole 2 secs. B will gain as much as A loses altogether.

89. The same considerations, when applied to the mutual reactions of any two bodies, show that one always gains as much momentum as the other loses, whatever the mechanism by which they exert force on one another. This may be shown experimentally in other cases, as follows.

Experiment ii. Put two trolleys on a plane, sloped to overcome frictional resistance; the trolleys should be provided with a stud of ivory or steel in the end of each, which act as buffers when the trolleys collide. Stand one trolley in the middle of the plane, and start the other with a push from the upper end of the plane; take a tracing of the motion of each, the frequency of vibration of the two springs being known, or at any rate being adjusted to be equal. Determine the uniform speed of the first trolley before it collides with the other, and the speeds of each after the blow; determine the masses of the trolleys, and calculate the momentum of each before and after the collision. It will be found that the "algebraic" sum of the momenta of the two trolleys after the collision will be very approximately equal to the momentum of the trolley originally set in motion. If the mass of the stationary trolley is a good deal greater than that of the moving trolley, the velocity of the latter may be reversed in direction by the collision, so we must use the "algebraic," not numerical, sum of the final momenta.

For example, the mass of the moving trolley is 7·75 lbs. and of the stationary trolley is 6·5 lbs., and the speed given to the former is observed to be 2·68 ft. per sec., the final speed of the former is ·975 ft. per sec. in the same direction as before, and the speed of the latter is 1·92 ft. per sec. Then the original momentum is **20·76** f.p.s. units, and the algebraic sum of the final momenta is $7·75 \times ·975 + 6·5 \times 1·92$ or **20** f.p.s. units.

Experiment iii. Repeat the experiment, using instead of trolleys with ivory buffers one furnished with a spike and the other with a cork into which the spike will stick when they collide; then after the collision the two trolleys will move on as one body. In this case, a tracing of the motion of the one trolley will be sufficient. As before, the final momentum will be found to be equal to the momentum before collision.

For example, the mass of the moving trolley is 7 lbs., that of the stationary trolley is 4 lbs., and the initial and final speeds are found to be ·6 and ·382 ft. per sec. respectively. Then the original momentum is 7 × ·6 or **4·2** f. p. s. units, and the final momentum is (7 + 4) × ·382 or **4·202** units.

Experiment iv. Two trolleys stand in a line, one carrying a pistol, the other a target; an electro-magnet is arranged on the first trolley so that it pulls the trigger when contact is made by means of a switch; the switch and battery are separate from the trolley, and the circuit is completed when the trolley is at its starting position through a pair of sliding contacts which do not impede the trolley's motion. If the pistol is fired so that the bullet enters the target, and tracings of the motion of the two trolleys are taken, it will be found that the backward momentum of the pistol and its trolley is equal to the forward momentum of the bullet and its trolley. If we know the mass of the bullet we can calculate its muzzle velocity, assuming that the momentum of the bullet is equal to that of the pistol and its trolley, or of the bullet, target and trolley; the bullet of course transfers this momentum to the target when it penetrates it.

For example, the trolley carrying the pistol and electro-magnet weighs 4·5 lbs., the trolley carrying the target weighs 7·2 lbs. After the shot has been fired the speed of the former is found to be ·78 ft. per sec., and of the latter ·5 ft. per sec. So the momentum of the former is **3·51** f. p. s. units, and of the latter **3·6** f. p. s. units.

If we take the mean of these two values as being the most probable value of the momentum, the momentum of the bullet must have been 3·55 f. p. s. units. The mass of the bullet was 1·75 grms. or ·003859 lb.; hence its speed was $\frac{3{\cdot}55}{{\cdot}003859}$ or 920 ft. per sec.

90. Conservation of Momentum. All these are illustrations of the general principle called The Conservation of Momentum. "*The total momentum of any system of bodies is* *not changed by any mutual actions between the bodies forming the system.*" We have shown that it must be the case by theoretical reasoning, and have verified it numerically in several instances ; it can however be deduced directly from the definition of Mass, and is not an important law of Nature like the law of Conservation of Energy. This latter law can only be established by a long series of experiments dealing with the various forms which energy can take, mechanical, thermal, electrical, etc., whereas the law of conservation of momentum requires no experiments to establish it ; but experiments are convenient for making clear the meaning of the law.

Ex. 25. A railway truck of mass 10 tons moving at 5 miles an hour hits another of mass 8 tons standing still, and is automatically coupled to it; with what speed will they run on ?

Ex. 26. A bullet of mass 1 oz. moving horizontally at 2400 ft. per sec. buries itself in a block of wood weighing 10 lbs., hung up so that it is free to swing horizontally ; what will be the inital speed of the block and bullet? Hence calculate the kinetic energy of the block and bullet. If the block is hung as in Fig. 66, so that it swings upwards parallel to itself, how high will it rise before coming to rest, i.e. before its kinetic energy is all changed into potential energy ?

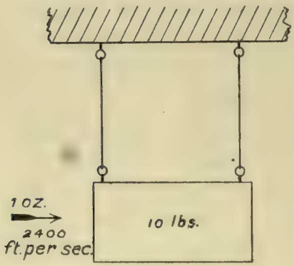


Fig. 66.

The initial momentum of the bullet is $\frac{1}{16} \times 2400$ f.p.s. units ; hence the velocity of the block and bullet is $\frac{1}{16} \times 2400 \div \left(10 + \frac{1}{16}\right)$ or $\frac{2400}{161}$ ft. per sec. Hence its kinetic energy is $\frac{1}{2} \times \frac{161}{16} \times \frac{2400}{161} \times \frac{2400}{161}$ ft.-poundals. When it has risen vertically through x ft., it will have gained $\frac{161}{16} \times 32{\cdot}2 \times x$ ft.-poundals of potential energy, so if it rises x ft. before coming to rest

$$\frac{161}{16} \times 32{\cdot}2 \times x = \frac{1}{2} \times \frac{161}{16} \times \frac{2400}{161} \times \frac{2400}{161},$$

whence
$$x = \frac{2400 \times 2400}{2 \times 32{\cdot}2 \times 161 \times 161} = \mathbf{3{\cdot}45} \text{ ft.}$$

This is a method used for measuring the muzzle velocity of bullets.

Ex. 27. If a bullet of mass 1 oz. is fired into a block of wood weighing 12 lbs., as in Ex. 26, and the block is observed to swing up through a vertical height of 2 ft., what was the muzzle velocity of the bullet?

91. Loss of kinetic energy in a collision. In nearly all cases of collision, although the total momentum is "conserved," that is, maintained without loss, the *kinetic energy* is not conserved, some of the energy being changed into heat. An example will show this best; take the case of Experiment iii in Art. 89. The initial kinetic energy is $\frac{1}{2} \times 7 \times \cdot 6^2$ ft.-poundals or **1·26** ft.-poundals; after the collision the kinetic energy is $\frac{1}{2} \times 7 \times \cdot 382^2 + \frac{1}{2} \times 4 \times \cdot 382^2$ ft.-poundals, or **·8026** ft.-poundals. So more than a third of the kinetic energy is lost in the collision. If the example given under Experiment ii is worked out in the same way, it will be found that the proportion of kinetic energy lost is smaller than this; the buffers in this case are more "elastic" than in Experiment iii, where they are entirely without elasticity; the steel spiral spring described in Experiment i is practically perfectly elastic, and if the values of the original and final kinetic energies are calculated, it will be found that there is no loss of kinetic energy.

But this is very exceptional; even with buffers made of ivory or steel (which are very elastic materials) there will be some loss of kinetic energy; the force which they exert while springing back into their original shape being less than that which was needed to distort them from that shape.

In choosing the method of attacking a given problem, this fact must be remembered, and unless you are quite certain that there is no change in the amount of *mechanical* energy, you must not use the principle of conservation of energy. When the energy changes its form from kinetic to potential (e.g. in the flight of a projectile) the principle can be applied, but not when mechanical energy is changed into electrical, thermal or other forms. In either case, the principle of conservation of momentum can be applied, but is frequently more roundabout than the energy method, which should therefore be used when possible.

Ex. 28. A coal truck of 10 tons weight is standing on the line when another truck of 7 tons weight runs into it at 8 miles an hour. The result of the collision is that the 7 ton truck stops dead. Find, by the principle of the conservation of momentum, the speed at which the 10 ton truck starts off. Hence calculate the loss of kinetic energy in the collision, expressed in ft.-poundals.

92. Collision with a "fixed" body. When a body (for example a racket ball) collides with another which is rigidly fixed to the earth (for example the wall of the racket court), the principle of the conservation of momentum must still hold. But now one of the bodies is the earth, and its mass is so enormous that a moderate change in its momentum will not involve any appreciable change in its velocity. So we must treat this body as immoveable, and the principle of conservation of momentum will not help us to find how the body moves in such a case.

93. Average force. When two bodies act on one another, the force between them is seldom constant, and often we do not know how it varies. In such cases it is sometimes convenient to know what was the average value of the force, i.e. *that constant force which, acting for the same time as the variable force, would produce the same change of momentum.*

This average value of the force is of course less than the greatest value to which the force actually rises if it is variable, but it is not usually less than half of that value, so it gives us some approximation to the force actually exerted.

For example, a ship of 10,000 tons moving at 3 miles an hour after being launched is stopped in 1 minute; what is the average value of the force? 3 miles an hour is 4·4 ft. per sec.; so the momentum destroyed is $10{,}000 \times 2240 \times 4{\cdot}4$ f. p. s. units. If F poundals be the constant force which acting for 60 secs. destroys this momentum, then $F \times 60 = 10{,}000 \times 2240 \times 4{\cdot}4$, whence $F =$ **1,643,000** poundals, or **22·8** tons weight. If the force stopping it were constant, the retardation would be uniform; its average speed would then be 2·2 ft. per sec., so in 60 secs. it would move 132 ft. If the force is not constant, we cannot calculate from the above data how far it will move before coming to rest.

Ex. 29. A hammer whose mass is $\frac{1}{2}$ lb. strikes a nail with a speed of 12 ft. per sec.; if its velocity is destroyed by the reaction, and if it is in contact with the nail for $\frac{1}{100}$ sec., find the average force of the blow in poundals and in lbs. wt.

Ex. 30. A hammer moving at a speed of 18 ft. per sec. strikes a nail and drives it $\frac{1}{2}$ inch into a board. Assuming that the resistance to penetration is constant, and that the mass of the nail may be neglected, find the time during which the force acts.

Since the retardation is constant, the average speed while the hammer's momentum is being destroyed is 9 ft. per sec.; since it moves ·5 inch with this average speed, the time occupied in doing so is $\frac{\cdot 5}{9 \times 12}$ or $\frac{1}{216}$ sec.

Ex. 31. If the mass of the hammer in Ex. 30 is 1 lb., find the average force of the blow. Hence find the work in ft.-poundals done during the blow. Compare this with the initial kinetic energy of the hammer.

Ex. 32. A cricket ball weighing $5\frac{1}{2}$ oz. is travelling horizontally at a speed of 20 ft. per sec., and is hit straight back with a speed of 60 ft. per sec.; find the change of momentum; if the bat was in contact with the ball for one-twentieth of a second, find the average value of the force exerted by the bat.

Ex. 33. Explain why a rubber-cored golf ball is less destructive than a "gutty" ball, to the face of the driver.

(Note that the former is in contact with the driver for a longer period than the latter.)

94. Average force during a succession of blows.

In most cases of collision the time during which the two bodies are in contact is so short that it requires elaborate experiments to measure it. But if a body is exposed to a very rapid succession of blows, each of which may last a very short time, but the whole succession lasting for some little time, then we can both measure and calculate the constant force which, acting for the whole of this time, would produce the same total effect, i.e., the "average" force.

For example, take the case of a Maxim gun which fires 1 oz. bullets with a muzzle velocity of 2000 ft. per sec. at the rate of 10 bullets a second; if these bullets strike a target they will act on it with a series of very large forces, each lasting a very short time, but with such short intervals between them (one-tenth of a second) that they would appear, to anyone holding up the target, to merge into one almost continuous pressure (somewhat in the same way as a series of pictures in a cinematograph leaves a continuous impression on the eye). We can easily calculate the value of this average pressure; for during each $\frac{1}{10}$th of a second, one bullet has its momentum destroyed, and this momentum is $\frac{1}{16} \times 2000$ f.p.s. units; so the constant force, F poundals, which, acting for one-tenth of a second, would destroy this momentum, is given by the equation $F \times \frac{1}{10} = \frac{1}{16} \times 2000$, whence $F =$ **1250** poundals, or **38·8** lbs. wt. This then is the average pressure on the supports of the target produced by the stream of bullets.

And since action and reaction are equal, this is the average pressure on the carriage of the Maxim gun; in this case the pressure is much more continuous than it is on the target, since the recoil at each shot is taken up by springs and the inertia of moving parts (which eject the empty cartridge and reload the gun for the next shot) so that the backward thrust of the carriage on the ground is smoothed out into an almost uniform force of about 39 lbs. wt.

Miscellaneous Exercises.

Ex. 1. A gun of mass 35 tons fires a shot of 850 lbs. with a velocity of 2000 ft. per sec. The gun recoils against a constant hydraulic pressure for ·125 second. Find the pressure in tons wt.

Ex. 2. A truck of mass 4 tons moving with a speed of 5 ft. per sec. runs into another of mass 10 tons moving with a speed of 1 ft. per sec. in the opposite direction. The former rebounds after impact with a speed of 1·5 ft. per sec.; find the velocity of the latter after impact. If the time of impact is one-fifth second, find the average force between them in tons wt.

Ex. 3. A hammer head of weight $2\frac{1}{2}$ lbs. moving at a speed of 50 ft. per sec. is stopped in ·01 sec. Find the average force of the blow.

Ex. 4. A man can do a standing jump of 10 ft. on level ground. If his weight is 12 stone, and he wishes to jump from a boat weighing 2 cwt. on to a pier at the same level, how far can the boat safely be from the pier, neglecting friction? (Find his actual horizontal velocity, assuming that the sum of this and of the speed of the boat is the same as his horizontal velocity when jumping from a fixed object.)

Ex. 5. A squash racket ball weighing $\frac{1}{2}$ oz. strikes the back wall at right angles to it with a speed of 100 ft. per sec. and rebounds with a speed of 80 ft. per sec., the time of contact with the wall being ·1 sec. Calculate the average pressure on the wall.

Ex. 6. A cricket ball, mass $5\frac{1}{4}$ oz., travelling at 64 ft. per sec. is caught and during the catch the hands are drawn back through 1 foot. Find the average force on the hands during the catch.

Ex. 7. A trolley, mass 10 lbs., moving at 12 ft. per sec. overtakes and sticks to another of mass 6 lbs. moving in the same direction at 4 ft. per sec. What will be the common speed after impact?

Ex. 8. A Maxim gun fires 300 half-ounce bullets per minute with a velocity of 1500 ft. per sec. Find the average force on the gun carriage.

Ex. 9. A shell bursts during its flight, when its speed is 1000 ft. per sec., breaking into two fragments whose weights are 8 and 24 lbs. Both travel onward in their original direction, the former at a speed of 1600 ft. per sec. Find the speed of the latter, from the fact that the total momentum after the explosion is equal to the momentum of the shell before it burst.

Ex. 10. In the case of the shell in Ex. 9, find the change of speed in each part of the shell and hence show that one gains as much momentum as the other loses.

Ex. 11. A shell weighing 32 lbs. buries itself in a box of earth of mass 1 ton, which is hung so that it can swing freely; the initial speed of the combined body is observed to be 20 ft. per sec.; calculate the speed of the shell.

Ex. 12. A body of mass 16 lbs. moving in a straight line has its speed reduced in 4 secs. from 32 to 24 ft. per sec. Find the change of momentum, and the average force in lbs. wt. opposing its motion during this time.

Ex. 13. A rifle bullet whose mass is 25 grms. strikes horizontally a block of wood whose mass is 2000 grms. lying on smooth ice, with a velocity of 400 metres per sec., and remains embedded in it. Find the velocity with which the block moves.

Ex. 14. A pile driver weighing 300 lbs. falls freely through a height of 10 ft. and is stopped in ·1 sec. Find the average force it exerts on the pile.

Ex. 15. A ball of mass $5\frac{1}{2}$ oz. strikes a bat with a speed of 20 ft. per sec., and the duration of contact with the bat is ·05 sec. Find the average force exerted to return it straight back to the bowler at a speed of 50 ft. per sec.; also the magnitude and direction of the average force required to send it at 50 ft. per sec. in the direction of Cover Point, that is at an angle of 120° with its original direction.

Ex. 16. A motor car weighs 30 cwt. and carries five people averaging 12 stone. It is moving at 20 miles an hour; what will be the average force needed to stop it in 10 secs.?

Ex. 17. A 12 stone man sits with his back to the engine in a railway carriage moving at 60 miles an hour; a collision occurs and his compartment is stopped in 1 sec.; find the average pressure on his back, in lbs. wt.

Ex. 18. The diameter of bore of a gun is 12 inches, and its length is 32 ft.; the projectile weighs 850 lbs. and its muzzle velocity is 2400 ft. per sec. Assuming that the acceleration of the projectile while in the gun is uniform, find the time taken to traverse the length of the gun; hence find the average force on the projectile and the average pressure per sq. in.

Ex. 19. If the gun in Ex. 18 is brought to rest in 18 ins., find the pressure in tons wt. exerted by the brakes.

Ex. 20. Steam is shut off a train when running on the level at 60 miles an hour, and the train slows down till it stops, the brakes not being put on. It is found to stop in 12 mins. 4 secs. after running 4 miles 863 yds. Show that the acceleration was not constant, and hence that the frictional resistances which stopped the train were not constant.

Ex. 21. Calculate the constant force which would bring a train of 200 tons, running on the level at 60 miles an hour, to rest in 724 secs.; and calculate the constant force which would bring it to rest in 23709 ft.

Ex. 22. Calculate the average values of the force for the train of Ex. 20 by the methods of Arts. 59 and 93, assuming the mass of the train to be M lbs.; is the time-average the same as the space-average in this case? Would these averages have been the same if the acceleration had been uniform?

Ex. 23. A ball of mass 1 oz. strikes a wall normally at a speed of 20 ft. per sec. and rebounds at a speed of 14 ft. per sec. Calculate the change of momentum and the loss of kinetic energy of the ball.

Ex. 24. A coal truck of mass 12 tons is standing on a level line when another truck of 8 tons mass runs into it at 8 miles an hour. The collision results in the 8 ton truck stopping dead. Find how far the other truck runs on before coming to rest, assuming that the frictional resistance to its motion is 10 lbs. wt. per ton.

Ex. 25. A body of mass 25 lbs. is moving due N. at a speed of 4 ft. per sec. A west wind exerts a force of 8 lbs. wt. on it for 4 secs. Find the final momentum.

Ex. 26. H.M.S. "Iron Duke" has a mass of 25,000 tons; she can fire a broadside of ten 13·5 inch guns, each firing a shell weighing 1450 lbs. with a muzzle velocity of 2000 ft. per sec. Calculate the speed, in miles per hour, with which she begins to recoil, neglecting the resistance of the water.

Ex. 27. Three goods trucks, each weighing 8 tons, and an engine weighing 40 tons stand on a level line with their buffers in contact; when their couplings are extended there is a gap of 2 ft. between each pair of buffers. The engine starts under a force of 500 lbs. wt., which remains constant. When any truck has been picked up, it moves on as one body with those in front of it. Calculate the speed of each truck just after it has been set in motion.

Ex. 28. A 9·2 inch gun, which weighs 28 tons 6 cwt., has its velocity of recoil destroyed in 18 ins. by an average force of 74 tons wt. The shell weighs 380 lbs.; find its muzzle velocity.

Ex. 29. A billiard ball, mass ·4 lb., strikes a cushion at an angle of 50° with the normal to the cushion, at a speed of 8 ft. per sec. It rebounds with a speed of 7 ft. per sec. at an angle of 45° with the normal; calculate the total change of momentum of the ball. Hence, if the time of contact with the cushion is one-fiftieth of a sec., find the magnitude and direction of the force exerted by the cushion on the ball.

Ex. 30. A railway carriage of mass 20 tons moving at 3 miles an hour runs into another of the same mass at rest. Spring buffers take up the kinetic energy lost in the first part of the impact until the carriages are moving forward as one body. Calculate this loss of kinetic energy. If each of the four buffers are then compressed through 3 inches, what is the force in lbs. wt. which each is then exerting?

CHAPTER VIII

FLUID PRESSURE ON A SURFACE

95. Pressure exerted by a jet of water. In Art. 94 we considered the average pressure exerted on the target by a stream of bullets which struck it at very short intervals.

A much more continuous thrust is produced by a stream of water striking against a fixed plate, and having its forward velocity thereby destroyed. We can use the same principle as before to calculate the value of the thrust. For example, suppose that a jet of water from a hose is directed at right angles to a wall, and that the water falls vertically downwards after striking the wall. Suppose that the area of cross-section of the jet is half a square inch, and that the speed of the water is 100 ft. per sec. Then the volume of water which reaches the wall in a second is that of a cylinder 100 ft. long and $\frac{1}{2}$ sq. inch or $\frac{1}{2 \times 144}$ sq. ft. in cross-section; so it is $100 \times \frac{1}{2 \times 144}$ or ·346 cub. ft. Since 1 cub. ft. of water may be taken to weigh 62·4 lbs., the mass of water which strikes the wall in one second is $\cdot 346 \times 62 \cdot 4$ or 21·56 lbs. The forward velocity of this mass is reduced from 100 ft. per sec. to zero; so the momentum destroyed in a second is $21 \cdot 56 \times 100$ or 2156 f.p.s. units. Hence if F poundals is the constant force which destroys it, we have $F \times 1 = 2156$ or the pressure on the wall is **2156** poundals, or **66·95** lbs. wt.

Ex. 1. A jet of water of cross-section $\frac{1}{4}$ sq. in. and velocity 50 ft. per sec. strikes a plate at right angles; find the pressure on the plate.

Ex. 2. A fire-hose delivers 800 gallons of water per minute at a speed of 60 ft. per sec.; find the pressure of water on a wall which it strikes at right angles. [1 gallon of water weighs 10 lbs.]

Ex. 3. Wind whose speed is 60 miles an hour blows at right angles to a wall; assuming that it loses its speed in this direction, find the pressure per sq. ft. of the wall; wt. of air per cub. ft. = ·0807 lb.

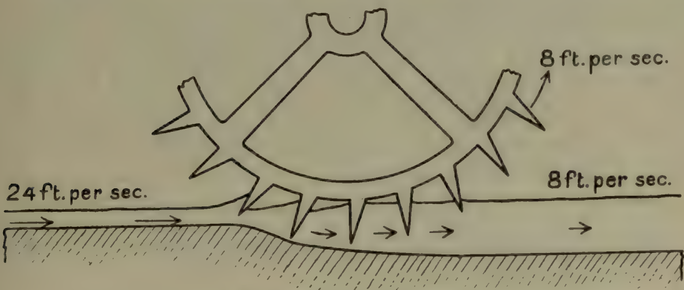
96. Pressure of a stream on a surface moving in same direction. The pressure produced by a stream of fluid striking a surface can be turned to practical use only when the surface on which it presses yields to the pressure; for example, the pressure of the wind on the sails of a vessel is only useful when the ship is urged forward by the pressure. We must take into account the velocity of the surface; in the simple case in which the stream of fluid is moving at right angles to the surface, and the surface moves in the same direction at a lower speed, we may consider that the "forward" speed of the fluid is reduced, on striking the surface, to that of the surface. Then the change of velocity in the direction of motion is equal to the difference of the speeds of the fluid and the surface; and the amount of the fluid which catches up the surface in any given time is the same as would be the case if the surface was at rest and the fluid moved towards it with this difference of the two speeds.

For example, if a wind of 30 miles an hour is blowing directly behind a motor car moving at 20 miles an hour, it would be the same to a passenger as if he were at rest and the wind were blowing in the same direction at 30 – 20, or 10 miles an hour. The air which strikes the back of the car moves on with a speed of 20 miles an hour after striking it, or loses 10 miles an hour (of course it also acquires a velocity sideways, but we can neglect that). In the course of an hour, a column of air 10 miles long will hit the back of the car, for at the end of the hour a column of air 30 miles long has passed the place from which the car started, and the car has only got 20 miles from its starting point. Hence the amount of momentum destroyed in an hour by the car is the same as if it stood still in a wind of 10 miles an hour; hence the force of the wind urging it forwards is the same. Of course if the surface is moving towards the stream of fluid we must take the sum, instead of the difference, of the speeds.

97. Action of wind on a sail. As an example, suppose a boat has the wind directly astern, blowing at 20 miles an hour, and that she is moving at 8 miles an hour in consequence of the pressure of the wind, and that her sail has an area of 200 sq. ft. set at right angles to the wind. Assuming that the wind which strikes the sail does not afterwards move faster forwards than the sail (we are not concerned with any speed it may acquire at right angles to its former direction), we have in one hour a column of air, 20 − 8 or 12 miles long and 200 sq. ft. in cross-section, whose speed is reduced from 20 to 8 miles an hour, or by 12 miles an hour. So in 1 sec., a column of air $\frac{12 \times 1760 \times 3}{60 \times 60}$ or 17·6 ft. long and 200 sq. ft. in cross-section has its speed reduced by $\frac{12}{60} \times 88$ ft. per sec. or 17·6 ft. per sec. Since the mass of 1 cub. ft. of air is ·0807 lb., the total mass of the column is 17·6 × 200 × ·0807 lb., and the momentum destroyed in a second is 17·6 × 200 × ·0807 × 17·6 or 5000 f.p.s. units. Hence (see Art. 85) the force on the sail is 5000 poundals, or **155** lbs. wt.

98. Undershot water-wheel. As a practical example of the use made of the destruction of the momentum of water, we will consider an ordinary undershot water-wheel, and will calculate roughly the force on it in a particular case.

Suppose a stream of water flows at a speed of 24 ft. per sec. in a rectangular gully, 2 ft. broad, and that it fills it to a depth of 1 ft. In the gully (the bottom of which is shaped as in the figure) is a wheel with radial paddles, against which the water strikes; after this the water runs away down the "tail-race." If the paddles nearly fit the gully, the water in the tail-race will have the same speed as the circumference of the wheel; suppose that the work the water-wheel is set to do allows it to move so that this speed is 8 ft. per sec. Then the depth of the water in the tail-race will be 3 ft., since it moves only at one-third of the speed of the water where it is 1 ft. deep (this may appear an unimportant detail, but it helps to make clear where the water goes when its speed is reduced by the paddles). In one second the volume of water which reaches the wheel is that of a column 24 ft. long and 2 sq. ft. in cross-section, or 24 × 2 or 48 cub. ft. Since 1 cub. ft. of water weighs 62·4 lbs., the mass of this is 48 × 62·4 lbs. or 299·5 lbs. This mass of water loses 24 − 8 or 16 ft. per sec. of its velocity. So the momentum destroyed per



Undershot water-wheel.

Fig. 66 *a*.

second is 299·5 × 16 f.p.s. units or 4792 f.p.s. units; so the force exerted on the wheel is 4792 poundals, or **148·5** lbs. wt.

It is important to note the difference between this example and the last; here we have a succession of surfaces against which the fluid strikes, instead of a single one, and the quantity of fluid which in a given time acts on the body to which the surfaces are attached is consequently greater, being unaffected by the speed of the individual surfaces because the body does not itself move forward as a whole.

Suppose now that the wheel does lighter work, so that the speed of the paddles is increased to 12 ft. per sec.; then the force along the circumference of the wheel can be calculated as before and will be found to be **111·4** lbs. wt.; so the force is reduced as the speed of the wheel increases. It would of course be zero if the paddles moved as fast as the oncoming water.

But it does not follow that the wheel is less efficient for its purpose when its speed is higher; we want the wheel to do work, and its "power," or the rate at which it does work, depends on the speed and force combined. It is obvious that the force is greatest when the speed is nothing, and that the speed is greatest when the force is nothing, but in neither case does the wheel do any work at all; so the power probably increases to a maximum at some speed and then decreases again to zero. This can be shown to be the case by calculating the power at various speeds; for example, at a speed of 8 ft. per sec. the power is $8 \times 148·5$ ft. lbs. per sec. or 2·16 H.P., and at 12 ft. per sec. it is $12 \times 111·4$ ft. lbs. per sec. or 2·25 H.P.; a curve should be plotted, having speeds as abscissae and horse-powers as ordinates, and it will be found that the maximum horse-power is obtained when the velocity of the water is halved on passing the wheel.

Ex. 4. Find the horse-power of the above wheel when the speed of the paddles is 16 ft. per sec.

*99. Efficiency of an undershot water-wheel.

We will next obtain a general expression for the power of a water-wheel of this kind. Call the speed of the water before reaching the wheel V ft. per sec., its area of cross-section A sq. ft., the speed of the paddles and of the water after passing them v ft. per sec. Then the volume reaching the wheel in 1 sec. is AV cub. ft., so its mass is $62·4 \times AV$ lbs.; hence its momentum is $62·4 \times AV \times V$ f.p.s. units before reaching the wheel and $62·4AVv$ f.p.s. units after passing it; hence the momentum destroyed in 1 sec. is $62·4AV(V-v)$ f.p.s. units, and the force on the wheel is $62·4AV(V-v)$ poundals. In 1 sec. the circumference of the wheel moves v ft., so the work done is $\mathbf{62·4AV(V-v)v}$ ft.-poundals in 1 sec.

The total kinetic energy possessed by the $62·4 \times AV$ lbs. of water before reaching the wheel was $\frac{1}{2} \times 62·4 \times AV \times V^2$ ft.-poundals, so the fraction of the energy extracted by the wheel from the water, or its "efficiency," is

$$\frac{62·4 \times AV(V-v)v}{\frac{1}{2}62·4 \times AV \times V^2} \text{ or } \frac{2(V-v)v}{V^2}.$$

In particular if $v=\frac{1}{2}V$, the efficiency becomes $\frac{1}{2}$; and this speed of the wheel gives the maximum efficiency. For let the speed of the wheel be $\frac{V}{2}+x$ ft. per sec., then the efficiency becomes

$$\frac{2\left[V-\left(\frac{V}{2}+x\right)\right]\left[\frac{V}{2}+x\right]}{V^2} \text{ or } \frac{2\left(\frac{V^2}{4}-x^2\right)}{V^2} \text{ or } \frac{1}{2}-\frac{2x^2}{V^2};$$

whether x is positive or negative this value is less than the value of the efficiency when x is zero (i.e. $\frac{1}{2}$).

So an undershot water-wheel cannot extract more than half of the energy which reaches it.

100. Pressure exerted by a stream striking a flat surface obliquely. If a stream of fluid, instead of striking the fixed surface at right angles to it as described in Art. 95,

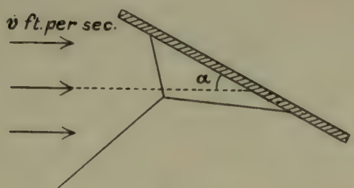


Fig. 67.

strikes it obliquely, we can calculate the pressure it exerts on a square foot of the surface by applying the same principles. An instance of this is a kite, held by its string in a wind; suppose Fig. 67 shows a section of it perpendicular to the plane of the kite. Call the inclination of the kite to the horizontal, $a°$; suppose the wind is blowing horizontally at v ft. per sec. Consider a particle of air striking the kite; after hitting it, we assume that it slides without friction along the surface, so that it loses the component of its speed perpendicular to the kite; and we are not concerned with changes in the component parallel to the surface, for these cannot produce any pressure on the surface. So the effect on the kite is the same as if it were in a wind blowing perpendicular to it at a speed equal to the component of the actual speed of the wind perpendicular to the kite, and the pressure on a square foot of the kite can be found as in Art. 95; it is $v \sin \alpha \times 1 \times \cdot 0807 \times v \sin \alpha$ or $\cdot 0807 \times v^2 \sin^2 \alpha$ poundals, since the wind's speed perpendicular to the kite is $v \sin \alpha$ ft. per sec.

This general expression is of use in giving a rough approximation to the pressure per square foot of a kite, the sail of a fore-and-aft rigged boat, or an aeroplane. In the case of the aeroplane, if it is travelling through still air, we merely substitute its speed for that of the wind in the above; for the only difference between it and the kite is that it generates the momentum perpendicular to its surface instead of destroying it. If there is a head wind and the aeroplane is flying horizontally, we must add together the speed of the aeroplane and of the wind to get v, and so on.

In order to find the lifting force per square foot of the plane we must take the vertical component of the pressure on a square foot, that is $\frac{\cdot 0807 \times v^2 \sin^2 \alpha}{g} \times \cos \alpha$ lbs. wt. The horizontal resistance to the motion is $\frac{\cdot 0807 \times v^2 \sin^3 \alpha}{g}$ lbs. wt. per sq. ft.

Ex. 5. Find a rough value for the lifting force per square foot of an aeroplane travelling horizontally at 75 miles an hour through the air, when the planes are set at an angle of 12° to the horizontal.

Ex. 6. If the total weight to be supported by the aeroplane is 1000 lbs., what must be the area of the planes?

Ex. 7. Find the horizontal force per sq. ft. needed to propel the aeroplane.

Ex. 8. Find the horizontal force needed to propel the aeroplane.

Ex. 9. Find the work done in propelling the aeroplane through 110 ft.

Ex. 10. Find the horse-power spent in propelling the aeroplane.

The expressions found in Arts. 95 and 100 for the wind pressures on a flat plane give values about twice those found in practice, because the air does not actually behave exactly as we assumed. The value of the pressure in lbs. per sq. ft. when the wind strikes the plane at right angles may with considerable accuracy be calculated from the formula $\cdot0029 \times V^2$, where V is the relative speed of the plane and air in miles per hour; when the plane is inclined at an angle α to the direction of the wind, the best formula for the pressure is

$$\cdot0029 \times V^2 \times \frac{2 \sin \alpha}{1 + \sin^2 \alpha}.$$

The corresponding formulae arrived at in Arts. 95 and 100 are $\cdot0054 \times V^2$ and $\cdot0054 \times V^2 \sin \alpha$.

The values found from either of these expressions in the last set of examples, for the lifting power, etc. of an aeroplane are fortunately not those necessary in practice, for the planes are made arched, like a bird's wing, and not flat. This enormously increases the lifting power per sq. ft. at a given speed, and reduces the power spent in propelling. For example, in a Blériot monoplane carrying a passenger, a weight of about 1000 lbs. is carried by an area of 260 sq. ft. (cf. Ex. 6), at a speed of only 42 miles an hour (cf. Ex. 5), with a 50 H.P. motor. But the calculations we have made serve to illustrate the principles and to show how we can arrive at rough values of the lifting capacity, etc. of an aeroplane by simple calculations.

101. The case of a sailing boat is not quite so simple. Let AB represent the sail, making an angle θ with the centre line of the boat and suppose that the boat is moving forwards at a speed of V ft. per sec. (we neglect any broadside movement, i.e. we assume there is no "leeway"). Suppose that the wind has a speed of v ft. per sec., and comes at an angle of α with the centre line, before the beam.

Then the speed of the sail in a direction perpendicular to itself is $V \sin \theta$ ft. per sec., and the component speed of the wind in this direction is $v \sin(\alpha - \theta)$ ft. per sec. Hence the pressure per sq. ft. of the sail is the same as if it were at rest in a wind blowing at right angles to it at a speed of $v \sin(\alpha - \theta) - V \sin \theta$

ft. per sec. Hence as before the pressure on a sq. ft. of the sail is $\cdot 0807 \{v \sin(\alpha - \theta) - V \sin\theta\}^2$ poundals.

The force per sq. ft. of sail area which is effective in driving the boat forward is the component of this force in that direction, or $\cdot 0807 \{v \sin(\alpha - \theta) - V \sin\theta\}^2 \sin\theta$ poundals.

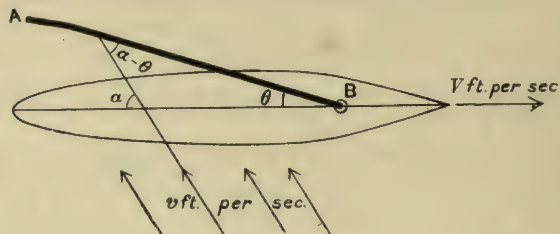


Fig. 68.

Ex. 11. If the wind is abeam (i.e. $\alpha = 90°$) and blowing at 5 miles an hour and the sail is set at an angle of 30° with the centre line of the boat, show that the wind will urge the boat forward even when the boat is travelling at 7 miles an hour, i.e. faster than the wind. (To prove this, it is only necessary to show that $v \sin(\alpha - \theta)$ is greater than $V \sin\theta$.)

Ex. 12. Calculate the force per sq. ft. of sail area which is effective in driving the boat forward, in the instance given in Ex. 11.

Ex. 13. Repeat Ex. 12, but taking the sail at an angle of 20° with the centre line of the boat.

Ex. 14. Repeat Ex. 12, for an angle of 35° with the centre line; which of these angles is the most effective?

102. Pressure of a jet of fluid on a curved surface.

Both steam and water turbines are actuated by the pressure produced by a jet of fluid on a solid surface as in an undershot water-wheel, but this surface is curved, not plane, and the jet of fluid is directed so that it slides on to the surface along a tangent to it, instead of striking it at an angle; the stream of fluid is afterwards deflected gradually by the shape of the surface. In this way "shock" is avoided and the momentum of the fluid is changed gradually and smoothly, thus preventing the formation of eddies, which waste energy.

For example, suppose a jet of water whose cross-section is 1 sq. in., moving at a speed of 40 ft. per sec., meets a fixed curved surface as shown in Fig. 69; it will retain its speed unchanged, if we neglect friction, because there is no force in the direction of its motion to change it; but the direction of that motion is changed.

Its velocity at exit can be resolved into two components (found by trigonometry or a scale drawing), one of 40 cos 60° or 20 ft. per sec. forward (i.e. in its original direction), and one of 40 sin 60° or 34·6 ft. per sec. perpendicular to its original

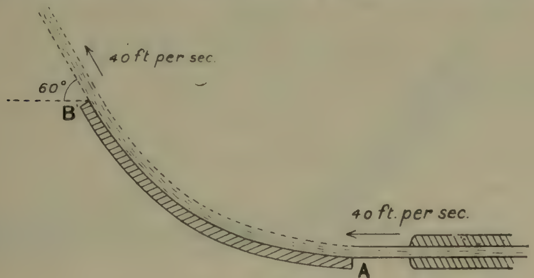


Fig. 69.

direction. So there is a change of velocity of 40 − 20, or 20 ft. per sec. in the former direction, and one of 34·6 ft. per sec. in the latter direction. Now the volume of water dealt with in 1 sec. is $\frac{1}{144} \times 40$ cub. ft., and the mass is $\frac{1}{144} \times 40 \times 62{\cdot}4$ or 17·3 lbs., so the changes of momentum are respectively $17{\cdot}3 \times 20$ f.p.s. units and $17{\cdot}3 \times 34{\cdot}6$ f.p.s. units. Hence the forces on the fixed surface are $17{\cdot}3 \times 20$ or **346** poundals in the direction of the jet and $17{\cdot}3 \times 34{\cdot}6$ or **598** poundals at right angles to it. From these we can if we wish calculate the magnitude and direction of the resultant force on the fixed surface.

As was pointed out in Art. 96, we can only make use of the pressure on the surface if the surface moves in the direction of the pressure. We will work out the same case, but will assume the surface to move with a speed of 15 ft. per sec. in the direction of the original jet.

When the water enters the curved surface at *A* it is moving relatively to the surface at a speed of 40 – 15, or 25 ft. per sec. Now imagine yourself to be standing on *AB*, and moving with it; the water at *A* will appear to you to move at a speed of 25 ft. per sec., and as the surface can exert no force on the water in the direction of flow (for we assume there is no friction) this speed cannot change; hence it flows along the surface at *B* at the same speed of 25 ft. per sec. But coming back again to the fixed surroundings of *AB*, we know that the point *B* is moving to

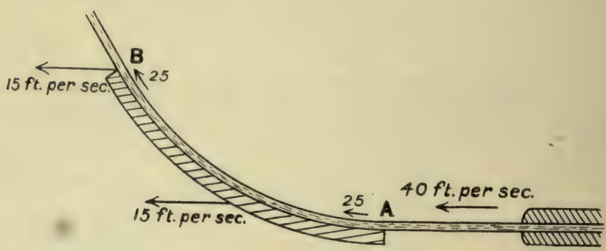


Fig. 70.

the left at a speed of 15 ft. per sec.; so the water as it emerges at *B* will be seen to have two velocities, one of 15 ft. per sec. in the original forward direction and one of 25 ft. per sec. in a direction inclined to it at 60°. Resolving the latter into components along and perpendicular to the original direction, we find these to be, respectively, 25 cos 60° or **12·5** ft. per sec., and 25 sin 60° or **21·65** ft. per sec. So the forward speed of the water at exit is 15 + 12·5 or **27·5** ft. per sec., and the speed at right angles to the original jet is 21·65 ft. per sec.

The quantity of water flowing on to the surface in one sec. is $\frac{1}{144} \times 25$ cub. ft., so its mass is $\frac{1}{144} \times 25 \times 62{\cdot}4$ or 10·82 lbs. Now the decrease of the water's velocity in the forward direction is 40 − 27·5 or 12·5 ft. per sec.; hence the change of momentum in that direction in 1 sec. is 10·82 × 12·5 or 135·4 ft. lb. sec. units. Hence the force on AB in the direction of motion is (see Art. 85) **135·4** poundals. Similarly the force at right angles to that direction is 10·82 × 21·65 or **234·5** poundals. These values should be compared with those found for the fixed surface.

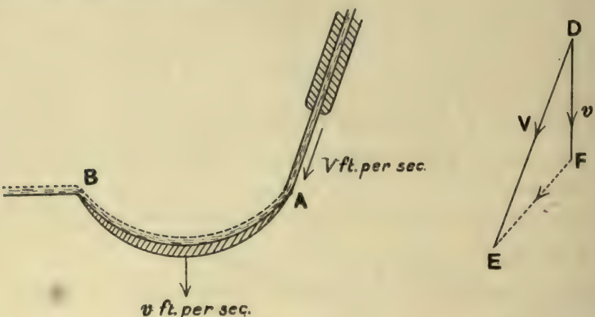
We will next find the actual velocity of the water at exit. We know that its components are 27·5 and 21·65 ft. per sec. at right angles to one another; compounding these we get for the magnitude of the resultant, 35 ft. per sec. Hence we see that when the surface moves, the velocity of the water changes its magnitude as well as its direction. This must clearly be the case, because the water exerts a pressure in the direction in which the surface moves, and therefore does work on the surface; this must extract kinetic energy from the water in the jet, which must therefore lose speed. In the above case it is easy to verify that the loss of kinetic energy equals the work done on the moving surface.

Ex. 15. Calculate the work done in one second on the surface in the case given above.

Ex. 16. Calculate the decrease of kinetic energy of the water which passes the surface in one second.

103. Use is made of the arrangement described in the last Article, in turbines, but in a modified form. It will be seen that no advantage was taken of the pressure on the surface at right angles to the direction of the jet; if the surface is permitted to move in that direction also, more of the energy of the incoming water would be extracted. But if the surface is allowed to move in this latter direction, and the jet remains as it is, the water will no longer enter the surface tangentially. This however can still be secured by pointing the jet in the direction of motion of the surface, more or less according to the speeds of the surface and the jet, as in Fig. 71.

Suppose that AB is free to move across the jet, as shown in Fig. 71 (i) with a velocity of v ft. per sec., and that the velocity of the jet is V ft. per sec. The velocity of the water relatively to AB is (see Art. 34) found by drawing the triangle of velocities as DEF in Fig. 71 (ii), "subtracting" v from V; FE represents in magnitude and direction the velocity with which the water appears to move as seen by a spectator stationed on AB and moving with it. Hence if the surface is shaped so that its tangent at A is parallel to FE the water will enter the surface tangentially, without shock.



(i) (ii)

Fig. 71.

AB is curved round backwards as shown, so that the momentum of the fluid in the direction of v is destroyed before it emerges at B; work is thereby done on AB, and a corresponding amount of kinetic energy is withdrawn from the fluid, which loses speed as well as changing its direction of motion.

But this will only be true for a very short time; as soon as the surface has moved a short distance, the jet will no longer strike the edge of the surface, but will reach it further inwards. This difficulty is overcome by arranging a series of such surfaces to pass the jet one after another, each of which receives the jet for a very short time and is then shielded by the one which follows. These surfaces (or "blades") are fixed on, and project radially from, the rim of a wheel, which as it rotates brings the surfaces in succession past the jet; the wheel is thereby urged forwards by the pressure on the blades attached to it. Fig. 72 gives a view of such an arrangement, looking at it in the plane of the wheel; the blades on the part of the rim immediately between the spectator and the axle of the wheel are seen end-on (since they are fixed radially); these are the only ones drawn in

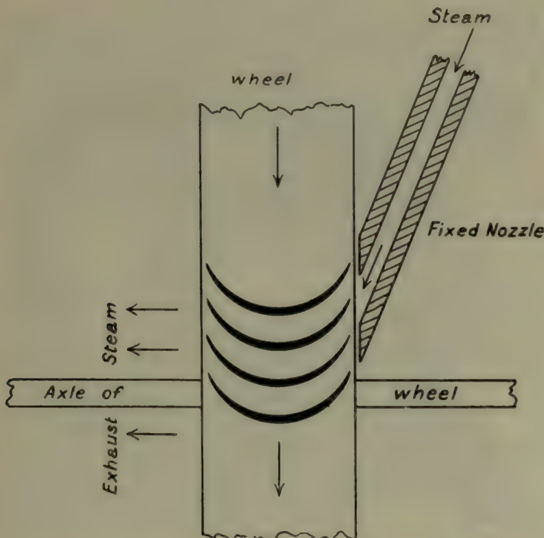


Fig. 72.

the figure, but of course there are similar blades all round the rim. If more power is required, additional jets may be fixed at other points of the rim.

104. De Laval Steam Impulse Turbine. Steam may be used instead of water in the turbine described in the last Article, and this form was invented by Dr De Laval. See Fig. 73. Owing to the small mass of a cubic foot of steam, it will have a very small momentum unless its speed is very great; and the force exerted on the wheel depends directly on the change of momentum of the fluid as it passes the blades. Now the proportion of the energy of the incoming fluid which is transferred to the wheel depends on the relation between the speed of the blades and the speed of the jet; the proportion is greatest when the former is about half the latter. Hence the

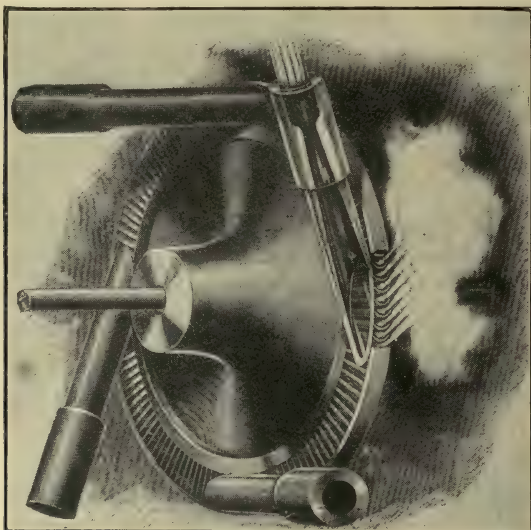


Fig. 73.

speed of the De Laval turbine must be very great if it is to be at all economical; in practice, it has to be geared down before it can be used even for driving dynamos, which run at high speeds.

Ex. 17. The speed of the steam in the jet is 3000 ft. per sec., and it is inclined at an angle of 20° to the direction of motion of the blades. The blades are so shaped that the steam on emerging from them moves at right angles to their direction of motion. Calculate the change of momentum, in the direction of motion of the blades, of one pound of steam.

Ex. 18. If $\frac{1}{200}$ lb. of steam flows from the jet in Ex. 17 in one second, determine the force exerted on the blades in the direction of motion.

Ex. 19. If the speed of the blades in Ex. 18 is 1000 ft. per sec., calculate the work done on the wheel per sec. in ft.-lbs., and hence the H.P. of the turbine.

Ex. 20. From the result of Ex. 19 calculate the loss of kinetic energy of $\frac{1}{200}$ lb. of steam. Hence determine the kinetic energy, and hence the speed, of the steam on leaving the blades.

Ex. 21. Calculate the kinetic energy of the steam passing the jet in one sec., and from the result of Ex. 20 deduce the efficiency (Art. 99) of the turbine. Compare it with that of the undershot water-wheel at its best.

Ex. 22. Repeat Exs. 19, 20 and 21 for a blade speed of 1500 ft. per sec.

105. Reaction of a jet of water on the pipe. When a shot is fired from a gun, the gun recoils; the force causing it to recoil is, by Newton's Third Law, equal to that which generates momentum in the shot. We saw in Art. 94 that there is a nearly continuous backward pressure, or reaction, on the carriage of a Maxim gun. We should expect to find that there is a similar continuous reaction on the pipe from which a jet of water is issuing, since momentum is being continuously generated. We can observe this reaction, and measure its amount, by the apparatus represented in Fig. 74.

A is a metal pipe which is bent at a right angle, and supported on knife edges at *B*, so that it is free to swing in a vertical plane; it is supplied with water by a long flexible rubber tube *C*, and the water issues in a horizontal jet. A spirit level is mounted on the horizontal part of *A* to show when the jet is horizontal. If the tube is hanging upright and the water is then turned on, it will swing backwards and an appreciable force will be needed to bring it back to its upright position and hold it there. This force can be determined by attaching a string to the tube, as shown in the diagram, and adding weights until the spirit level shows that the jet is once more horizontal; then the moment about *B* of the tension of the string and of the reaction of the jet must be equal, so the latter can be calculated when the distances from B at which these forces act have been measured. The mass of water which issues from the jet in a second can be determined by catching the water in a vessel and weighing it. After issuing from the nozzle the water follows the same course as a particle projected horizontally, so its horizontal velocity of projection can readily be determined by measuring the vertical height it has fallen in a definite horizontal distance. From these measurements we can determine

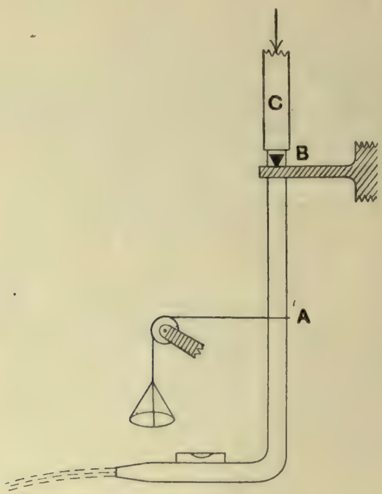


Fig. 74.

the *horizontal* momentum generated in a second; while the water is passing through the vertical part of A it has of course no horizontal momentum. What the metal tube A actually does is to destroy the vertical momentum of the water; to do this a force is needed, but this force acts through the knife edges at B, and so does not tend to turn the tube round B; it also generates a horizontal momentum in the water, and this needs a force acting along the horizontal part of the metal tube, which is what we have been measuring. The effect on the tube is similar to that described in Art. 104.

So a fireman holding the nozzle of a fire-hose has to exert a force to prevent the nozzle from moving backwards in consequence of the reaction at the first bend in the hose.

Ex. 23. In the above experiment the following rough measurements were made; 8 lbs. of water were collected in 15 secs., and the water was observed to fall through a vertical distance of 21 inches in a horizontal distance of 11 feet. The distance of the jet from *B* was 46 ins. and of the string from *B* was 32 ins.; the weight in the pan was ·81 lb. wt.

If we denote the horizontal velocity of the water by v ft. per sec., then the time in which it travels a horizontal distance of 11 ft. is $\frac{11}{v}$ secs.; so it falls vertically through 21 ins. or 1·75 ft. in $\frac{11}{v}$ secs. Its vertical velocity at the end of $\frac{11}{v}$ secs. is $\frac{11}{v} \times 32{\cdot}2$ ft. per sec., so its average vertical velocity is $\frac{11}{2v} \times 32{\cdot}2$ ft. per sec.; hence in $\frac{11}{v}$ secs. it will fall through $\frac{11}{2v} \times 32{\cdot}2 \times \frac{11}{v}$ ft. Hence we have $\frac{11}{2v} \times 32{\cdot}2 \times \frac{11}{v} = 1{\cdot}75$ or $v^2 = \frac{11 \times 32{\cdot}2 \times 11}{2 \times 1{\cdot}75}$ or $v = 33{\cdot}4$ ft. per sec.

Now the mass of water which in each second receives this horizontal velocity is $\frac{8}{15}$, or ·533 lb.; so the horizontal momentum generated in 1 sec. is ·533 × 33·4 f.p.s. units, or 17·8 f.p.s. units. Hence the reaction should be **17·8** poundals, or **·553** lb. wt.

The moment about *B* of the force (·81 lb. wt.) along the string is $\cdot 81 \times \frac{32}{12}$ lbs. wt. ft.; if F lbs. wt. be the reaction of the jet on the tube, the moment about *B* of this force is $F \times \frac{46}{12}$ lbs. wt. ft.; hence $F \times \frac{46}{12} = \cdot 81 \times \frac{32}{12}$, or $F =$ **·563** lb. wt. This value of the reaction is approximately what was found by considering the rate at which momentum was generated.

106. Reaction Motors. We can now understand the working of sundry toys, which date back to early Egyptian times, for producing motion by the reaction of issuing jets of fluid. Fig. 75 gives a plan of the simplest of these; *A* is a can of water, free to turn on a central vertical axle; tubes *B*, *B* emerge horizontally from the can, and a hole is made as shown in each tube. From these holes jets of water emerge, and the reaction drives the can round on its axle. It will be seen that the jets need not strike any fixed object; for the driving force is caused by the generation of momentum in the water. A similar effect will be produced by allowing steam instead of water to escape from the holes. It is a similar reaction which causes a rocket to fly.

Another toy depending on the same principle is a model boat containing a boiler; the steam is taken in a pipe to the stern and there discharged horizontally backwards from a nozzle. The reaction drives the boat forward. Here we have a store of steam in the boiler at high pressure; the "pressure energy" of this steam is allowed to convert itself into kinetic energy by expanding freely at the nozzle; so a certain amount of backward

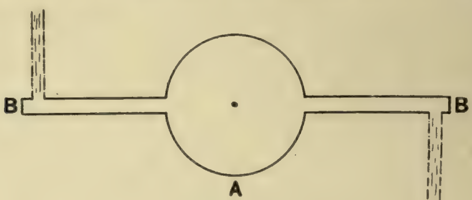


Fig. 75.

momentum is continually created *within the moveable vessel*, with the consequence that there is a force which drives the vessel forwards.

It should be noted that in all these cases the speed with which the fluid leaves the vessel must be greater than the backward speed of the vessel itself; for otherwise there would be no actual momentum generated relatively to the body which is resisting the vessel's motion, so no force of reaction would be available to overcome this resistance.

107. Propulsion of Ships and Aeroplanes. Advantage is taken of this reaction when we wish to make a body, such as a ship, move forward in the fluid in which it floats. A railway train is propelled by the friction between the rails and the driving wheels, but friction is not used in propelling ships; the friction of the water is merely a hindrance, not a help. Speaking broadly, the ship seizes masses of water from the still water near it and hurls them backwards; the production of this backward momentum causes a reaction which drives the ship forward, not because the water thus set in motion afterwards hits against a fixed body or presses against the surrounding water, but because the mere setting in motion of this water by itself produces a reaction on the body which has set it in motion. The same applies to aeroplanes, substituting air for water.

The method of propelling a ship which is theoretically the most perfec' consists in taking in water through a wide orifice at the bows of the ship, passing it through a pump driven by the

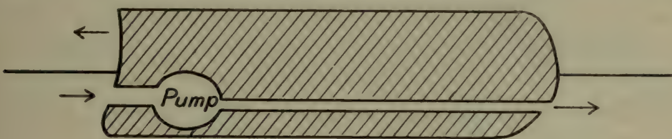


Fig. 76.

engines, which gives it considerable speed, and sending the narrow jet of water so produced directly astern, either above or below water level. This has been done on a large scale for experimental purposes, but did not prove so efficient as propellers; H.M.S. "Waterwitch," a vessel of 1160 tons; was fitted with pumps employing about 700 H.P., delivering roughly 150 cub. ft. of water per second, and attained a speed of 9·3 knots. The chief practical difficulty is that pumps of a reasonable size cannot deal with a very large body of water in a second, so that to produce the necessary momentum the speed given to the water must be high, and water at high speed easily loses energy in eddies; a similar vessel, with the same H.P. and displacement, fitted with twin screws, acted on more than 2000 cub. ft. of water per second, and could thus produce the same reaction by giving to the water a much smaller speed.

Ex. 24. In H.M.S. "Waterwitch" the total area of cross-section of the jets by which the water was discharged was 6·28 sq. ft. The ship moved at a speed of 9·3 knots, or 15·7 ft. per sec. Assume that the water enters the inlet orifices without acquiring any speed, i.e. at a speed of 15·7 ft. per sec. relatively to the ship; and that the pumps increase this speed to 29 ft. per sec. relative to the ship. Calculate the volume of water issuing in 1 sec. from the orifices at the stern; hence find the change of momentum in the water and the propelling force on the ship.

Calculate also the area of cross-section of the pipe leading from the inlet orifices to the pumps.

108. Both paddle wheels and screw propellers act by giving a backward momentum to water which was previously at rest, and the momentum generated in a second, expressed in foot-pound-second units, is equal to the force in poundals with which the ship is urged forwards. The stream of water driven sternward from the propeller is called "the race of the propeller"; it is roughly speaking a cylindrical column of water, of the same diameter as the propeller, moving backwards through the undisturbed water. The propeller does not "screw" its way forwards through the water as if it were solid, nor do paddle wheels act like the wheels of a traction engine; it is only by "slipping" in the water (and thus departing from the way it would behave if the water were a solid) that the propeller, or paddle wheel, can project water backwards and so urge the ship forwards. Similarly if the propeller is situated in the front of an aeroplane, the pilot sits in a stream of air which moves past him more rapidly than does the undisturbed air through which the aeroplane is travelling. He experiences the same effect, but to a smaller extent, if the propeller is behind him, for air flows in from all directions to take the place of that which is projected backwards by the propeller.

We can calculate roughly the value of the force urging the ship forwards, and the work required, when we know the dimensions and speed of the race and the speed of the ship through the water; but we must make many assumptions that cannot be realised in practice, so our result will only be an ideal to which the shipbuilder may aspire. Call the area of cross-section of the race of the propeller, A sq. ft., the speed of the ship V ft. per sec., and the speed of the race *relatively to the ship* v ft. per sec.; i.e. to a passenger the undisturbed sea appears to be moving backwards at V ft. per sec., and the water in the wake of the ship at v ft. per sec. The actual speed of the water in the race will be $v-V$ ft. per sec. Then in 1 sec. a volume of water Av cub. ft. is acted on by the propeller, for the cylinder of water which leaves it in a second is v ft. long. The mass of this water is $64 \times Av$ lbs., since 1 cub. ft. of sea-water weighs 64 lbs.; the velocity given to it by the propeller is $v-V$ ft. per sec., so the momentum generated per sec. will be $64 \times Av(v-V)$ f.p.s. units; hence the reaction on the ship is $64 \times Av(v-V)$ poundals or $\frac{64}{32\cdot 2} \times Av(v-V)$ lbs. wt., or $2Av(v-V)$ lbs. wt. very nearly.

It may be of interest to note that in ordinary steamers A is about one-third of the immersed midship section of the ship.

Ex. 25. To get some idea of what these speeds and pressures are in practice, take the following rough figures for H.M.S. "Drake," which is a twin-screw cruiser of 14,000 tons. She has a beam of 71 ft. and draws 28 ft. fully loaded; each of her propellers may be taken as having a radius of 9 ft. (the actual radius is 9 ft. 6 in., but the "boss" in the centre makes part of this useless). Hence A, the area of cross-section of the race of the two propellers together may be taken as $2 \times \pi \times 9^2$ sq. ft., or roughly 500 sq. ft. It is found by observation and calculation that to tow her at 10 knots requires a force of 3200 lbs. wt., and at 20 knots a force of 25,200 lbs. wt.; these then are the thrusts of the propellers when she is steaming at these speeds. (A knot is a speed of one sea-mile, or 6080 ft., in an hour.)

A speed of 10 knots is $\frac{6080 \times 10}{60 \times 60}$ ft. per sec., or **17** ft. per sec. nearly; so 20 knots is 34 ft. per sec.

If we substitute these numbers for A and V in the expression for the reaction on the propeller, we get

$$3200 = 2 \times 500v\,(v-17),$$

whence we find $v = 17\cdot 18$ ft. per sec. when the speed of the ship is 10 knots. So the "race of the propeller" only moves at a speed of $17\cdot 18 - 17$, or $\cdot 18$ ft. per sec., or about an eighth of a mile an hour, backwards through the undisturbed water.

At a speed of 20 knots we have $25{,}200 = 2 \times 500v\,(v - 34)$, whence we find $v = 34{\cdot}73$ ft. per sec. Hence, to produce this speed in the ship the race of the propeller moves through the undisturbed water at a speed of $34{\cdot}73 - 34$ or $\cdot 73$ ft. per sec., or four times as fast as before.

These speeds should be contrasted with the corresponding speed of the "race" in H.M.S. "Waterwitch," which for 9·3 knots was 13·3 ft. per sec. (see Ex. 24).

Ex. 26. Taking the aeroplane of Ex. 5 in Art. 100, and assuming that the propeller has a radius of 3 ft., we have $A = \pi \times 3^2$ or 28·27 sq. ft., $V = 110$ ft. per sec. Since 1 cub. ft. of air weighs ·0807 lb., we get as above, since the required horizontal thrust is 212·7 lbs. wt.

$$\frac{\cdot 0807}{32{\cdot}2} \times 28{\cdot}27v\,(v - 110) = 212{\cdot}7.$$

From this $v = 132{\cdot}6$ ft. per sec. So the wake of the propeller has to move backwards through the still air at about 15 miles an hour.

***109. Parsons' Steam Turbine.** The Parsons turbine is now so widely used that it is important to understand its

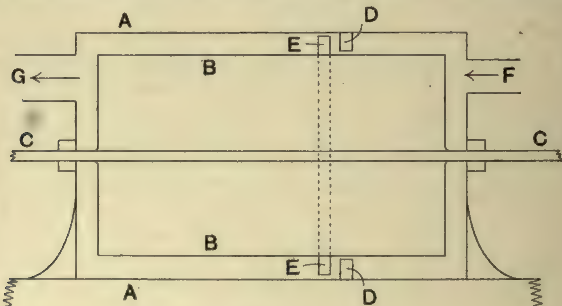


Fig. 77.

principle of action, at any rate in rough outline; and it forms a very good illustration of the reaction caused by the generation of momentum.

Broadly speaking, it consists of a fixed cylindrical casing *A* (Fig. 77); into one end steam is admitted from the boiler, and from the other end the steam passes away to the "exhaust" after it has lost its high pressure. Inside this casing and concentric with it is a drum *B*, whose diameter is about 6 inches less than *A*'s; this drum is mounted on a central shaft *C*, with which it can turn. The shaft conveys the power to the dynamo or propeller etc. which the turbine has to drive. The steam flows from one end, *F*, of the turbine to the other, *G*, along the space between drum and casing, and in so doing has to work its way among a number of blades which reach nearly across this space. The blades are arranged in parallel rows, fixed alternately to the casing and the drum; the plane of each row is perpendicular to the axle; each blade in one row *D* is fixed by one end to the casing and projects radially inwards, nearly touching the drum; the blades in the next row *E* are fixed to the drum, and project radially outward, nearly touching the casing. The blades in any one row are spaced apart at a small distance, thus providing channels between them by which the steam can pass the row of blades; if we imagine the casing to be transparent and look through it towards the central shaft, we shall be looking end on along the blades, and those two successive rows will appear somewhat as shown in Fig. 79. Only a few are shown here, but each row extends round the whole circumference.

Fig. 78 is a general view of a turbine, with the upper part of the casing removed.

When the turbine is running and doing work, the pressure of the steam drops as it passes through each row of blades, falling step by step from the high pressure at the boiler end to the low pressure of the exhaust. For simplicity we will assume that the turbine is so designed that there is a drop of pressure at each row of 1 lb. per sq. in. Consider what happens to the steam in passing through the channels between the blades of a row fixed to the casing. Suppose that on coming to this row it is at a pressure of 100 lbs. per sq. in. On emerging from the channels, the pressure is only 99 lbs. per sq. in.; this excess of pressure behind causes an acceleration of the steam in the channels, which will emerge at a speed about half as fast again as on entering these channels. In addition to this change of magnitude, the direction of the velocity has been gradually deflected, owing to the shape of the channels (see Fig. 79) until it is inclined at a small angle (say 20°) to the face of the row of blades. Each

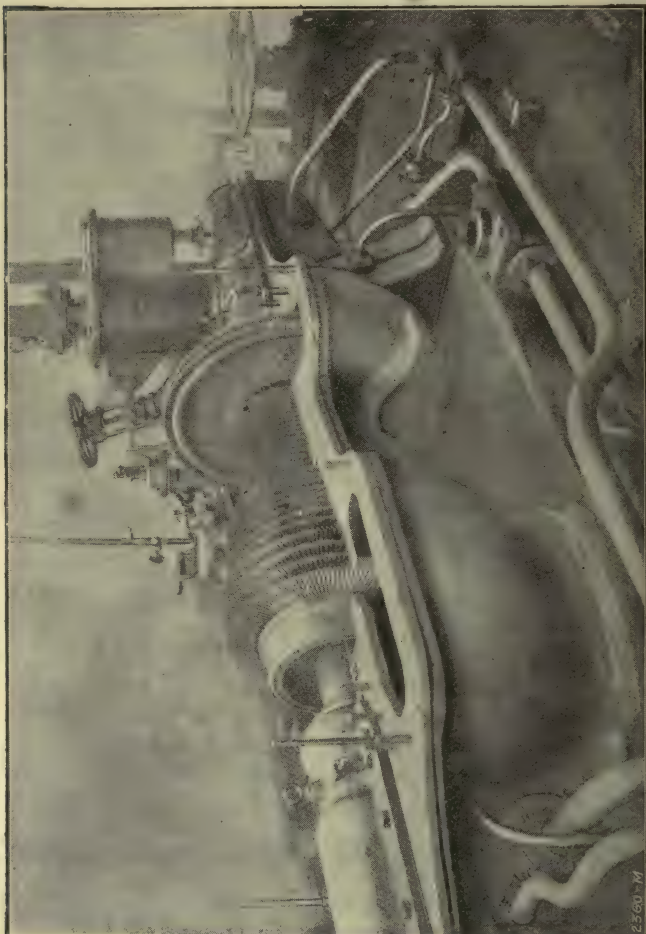


Fig. 78. Compound Impulse-Reaction Turbine.

of these channels therefore acts like a jet of the de Laval turbine (p. 184).

It now enters the next row of blades, which is fixed to the drum and therefore moves with it; as in the de Laval turbine this moving row of blades extracts some of the kinetic energy which the steam has gained in passing through the fixed row. But the channels in this moving row are curved backwards at their exit end more than at their entrance end (compare Fig. 79 with Fig. 72). The consequence of this is that there is a greater change in the direction of motion of the steam than in the de Laval turbine; instead of the steam emerging at right angles to the face of the row, it is directed backwards. Thus there is a greater change in the momentum of the steam and consequently a greater force of reaction on the moving blades. Hence more work is done on the blades, and so more energy is drawn from the steam. This would reduce the speed of the steam too much, but for the difference of pressure between the faces of the moving row, which causes an acceleration of the steam in these channels. The shape of the blades is arranged so that this drop of pressure shall bring up the speed to exactly what it was on entering the preceding fixed row.

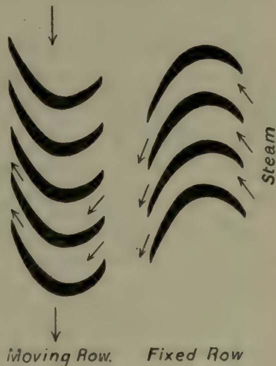


Fig. 79.

Another row of blades fixed to the casing is arranged immediately after the moving row, to take the steam as it emerges from it; then another row of blades fixed to the drum, and so on. Each pair of rows passes the steam on to the next, with a drop of pressure of, say, 2 lbs. to the sq. in., but moving with a velocity the same in magnitude and direction as it had on entering the first pair, until finally the steam emerges from the turbine at a low pressure, having surrendered nearly all its energy to the drum.

***110.** Thus although there is a perfectly free channel for the steam from one end of the turbine to the other, the direction of flow of the steam is continually being changed, and as the speed is high this involves a considerable change of momentum; consequently there is a reaction on the walls of the passages in which the direction of flow is changed, as in the pipe in Art. 105. So the steam does work on the moving drum of the turbine as much as if it were shut up in the cylinder of a reciprocating engine, where it does work by forcing out the piston. We may regard it in the engine as using its high pressure by thrusting against the piston, getting its "push off" from the solid end of the cylinder, and in the turbine as using its high pressure to get up speed in itself, having no piston or other impediment to check its forward speed, and getting its "push off" from the wall of the curved channel in which it is flowing. When it gets up this speed in the channels of a fixed row, the reaction on the wall is of no service, since it is fixed; but it carries its kinetic energy on to the moving row and surrenders it there, by having its momentum destroyed in the impact against the blades; this drives them forward as in an impulse turbine. When it gets up speed within the channels of a moving row, since these channels are directed backwards, the momentum generated is directed backwards and the reaction on the wall of the channel is directed forwards. Hence the moving blades are driven forwards partly by impulse and partly by reaction, and the Parsons turbine is classed as an Impulse-Reaction Turbine.

We may compare the moving blades of an impulse turbine to the sails of a ship, the high-speed steam acting as wind; and those of a reaction turbine to a ship propelled by pumps as in H.M.S. "Waterwitch," or propellers, which take the water and give it a backward momentum, thereby forcing the ship forwards by reaction. The latter analogy is not perfect, for the water has to be accelerated by the ship's engines whereas the steam accelerates itself by its own high pressure, but the principle is the same. Hence we may regard a moving row of blades of a Parsons turbine as a ship under both sail and steam.

Miscellaneous Exercises.

Ex. 1. A jet discharges 100 lbs. of water a minute against a wall, at right angles to it, at a speed of 50 ft. per sec. How much momentum is destroyed per minute by the wall, and what is the average force on the wall?

Ex. 2. A stream is flowing at 3 miles an hour at right angles to a floating bridge, which opposes an area of 120 sq. ft. to the stream. Find the force on the bridge exerted by the stream.

[1 cub. ft. of water weighs 62·4 lbs.]

Ex. 3. A vertical cricket screen measures 20 ft. by 12 ft.; a wind blows directly against it with a speed of 30 miles an hour. Taking the weight of 1 cub. ft. of air as ·08 lb., find the total pressure on the screen.

Ex. 4. The pier of a bridge has a vertical face at right angles to the direction of the stream; the breadth of the pier face is 10 ft. and the depth of the water is 15 ft., the speed of the stream being 30 ft. per minute. Find the force on the pier face.

Ex. 5. A ship runs at a speed of 4 miles an hour before a wind of 15 miles an hour; find the pressure on a sail whose area is 400 sq. ft., set square to the wind.

Ex. 6. The area of surface which a motor car opposes to the wind is 20 sq. ft. Calculate the wind resistance when running at 30 miles an hour on a calm day, and the H.P. needed to overcome it.

Ex. 7. If the propellers of the "Drake" (Ex. 25, p. 191) have a radius of 8 feet, calculate the speed of the race of the propeller for a ship speed of 10 knots.

Ex. 8. A jet of water $\frac{3}{4}$ inch in diameter enters tangentially the concave surface of a curved plate and is deflected through an angle of 40° by the plate, which is moving in the direction of the jet at a speed of 30 ft. per sec. The velocity of the jet is 60 ft. per sec. Find the total force exerted on the plate, in magnitude and direction.

Ex. 9. An aeroplane is being driven horizontally by a motor whose indicated horse power is 30. The efficiency of the motor and propeller is 50 per cent. The pressure of the air on the wings is a steady force of 800 lbs. wt. inclined upwards at an angle of 84° to the direction of motion. Calculate the weight of the aeroplane and the speed in miles an hour.

Ex. 10. A ship steams at 24 knots against a head wind which is blowing at 15 knots (1 knot = 6080 ft. per hour). The front of the chart house, which is a vertical plane perpendicular to the direction of motion, has an area of 90 sq. ft. Find the force exerted by the wind on the chart house, (i) assuming that the relative velocity of the air is completely destroyed by the impact, and (ii) by the formula of Art. 100.

Ex. 11. Water flows with uniform speed of 10 ft. per sec. through a pipe, whose diameter is $\frac{1}{4}$ inch, with a bend in it; the two parts of the pipe make an angle of 120° at the bend. Calculate the magnitude and direction of the reaction on the pipe.

Ex. 12. Water issues from a pipe at a speed of 80 ft. per sec. The pipe has a cross-sectional area of ·5 sq. ft. The water strikes normally against the blades of an undershot water wheel which are moving at 30 ft. per sec., and its velocity is reduced to that of the blades. What H.P. does the wheel exert?

Ex. 13. A kite has an area of 4 sq. ft., and flies at a slope of 30° to the horizontal, in a wind of 15 miles an hour. Using the formula of Art. 100, calculate the normal pressure of the wind on it. If the weight of the kite is 1 lb., determine the angle at which the string is inclined to the kite, and the tension of the string there.

Ex. 14. A canvas fire-hose is lying on the ground, not in a straight line, with the nozzle free to move. When the water is turned on, will the reaction at the bends tend to straighten out the hose? Give reasons for your answer.

Ex. 15. An approximate formula for the resistance to motion of a train is $5+\frac{V^2}{200}$ lbs. wt. per ton, where V is the speed in miles per hour. Calculate the effective area of the surface opposed to the wind, in the case of a train weighing 100 tons, to produce the second term in this formula. (Use the formula of Art. 100.)

Ex. 16. The propeller of a Deperdussin monoplane transmits 80 H.P., and its diameter is 9 ft. The speed of the aeroplane is 68 miles an hour. Calculate the thrust of the propeller; hence calculate the speed of the "wake" relatively to the airman (mass of 1 cub. ft. of air = ·08 lb.).

Ex. 17. The main-topsail of the "Victory" had an area of 4500 sq. ft. Calculate the force exerted on it by a wind blowing at 12 miles an hour when the ship was running before it at 6 miles an hour; hence calculate the horse-power of this sail (use the formula of Art. 100).

*CHAPTER IX

MOTION UNDER VARYING FORCES

111. Motion under varying forces. Forces that are absolutely constant in magnitude and direction seldom occur in everyday practice; we frequently find them constant in direction but varying in magnitude, and often their magnitude does not vary according to any simple law. In such cases we can usually observe the effects of the force (for example, the distances moved by the body at the end of successive intervals of time) and from the curves representing these results we can deduce a good deal of information about the motion and the force in action.

For instance, suppose that a body is in motion under the action of an unknown force, whose magnitude is changing, and that a series of observations of the distance of the body from a fixed point in its track at intervals of say 10 seconds gives the distance-time curve shown in Fig. 80.

By comparing this with the distance-time curves found for motion under a constant force (e.g. Art. 14), it is obvious that the force was not constant; and as the distance of the body from the starting point begins to decrease after about a minute, it is clear that the velocity does not increase all the time, but that the body stops and acquires a velocity in the opposite direction.

112. Calculation of velocity from a distance-time curve. We can determine the velocity of a body at any instant from the distance-time curve of its motion, by drawing a tangent to that curve at the point corresponding to that instant. For if we examine Fig. 10 on page 25, it hardly needs a formal proof to convince us that if the resultant force on the body suddenly ceases to act at any instant, the distance-time curve of the subsequent motion is the tangent to the actual distance-time curve of the body at that instant. Let us for the moment assume that this is so; a formal proof will be given presently. When the body is moving with constant velocity (as it does when the resultant force has stopped acting) its distance-time curve is a straight line and the velocity is given by the gradient of this straight line. (See Art. 7.) But this constant velocity is the

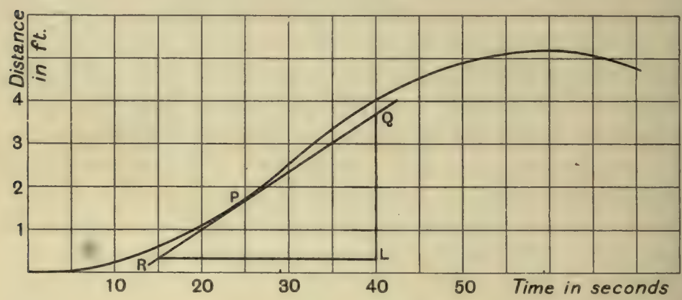


Fig. 80.

velocity at the instant when the force ceased to act; so the velocity at any instant during the motion under the force is given by the gradient of the tangent to the distance-time curve at the point corresponding to that instant.

For example, to find the velocity at a time 24 secs. in the case illustrated in Fig. 80, we draw the tangent at the point P corresponding to 24 secs., and measure the gradient of this line by drawing the lines QL, RL. QL represents 3·3 ft., and RL represents 25 secs.; hence the velocity required is $\frac{3{\cdot}3}{25}$ ft. per sec. or ·132 ft. per sec.

By determining in this way the velocity at several instants we can draw a velocity-time curve for the motion. But it will need great skill in drawing the tangents to a given curve, so that the results may produce a smooth and accurate velocity-time curve.

If the tangent in one part of the distance-time curve slopes downwards from right to left, and in another part it slopes downwards from left to right, it is obvious that the corresponding velocities have different signs, and will be represented on the velocity-time curve by ordinates on different sides of the horizontal time-axis.

Ex. 1. The following table gives the distances of a piston from its starting point and the corresponding times. Plot the distance-time curve (to scales 1 inch to ·5 ft. and 1 inch to ·1 sec.), and from it deduce the velocity at ·05, ·10, ·15, ·27, ·30, ·35, ·45 sec.

t in sec.	0	·05	·10	·15	·20	·25	·30	·35	·40	·45	·50
s in ft.	0	·13	·42	·8	1·2	1·56	1·83	1·98	1·99	1·87	1·64

Ex. 2. Plot the velocity-time curve in Ex. 1, on a scale of 1 in. to 5 ft. per sec., and 1 in. to ·1 sec. (Keep this curve for future use.) From it determine the instants at which the piston is at rest, and at which it is moving at greatest speed. Find from the distance-time curve whether this last instant is when the piston is at the mid-point of its stroke.

Ex. 3. The distances travelled up a gun by the projectile, and the corresponding times are as follows:

t in sec.	0	·001	·002	·003	·004	·005
s in ft.	0	·15	·8	1·9	3·3	5

Plot the displacement-time curve, and determine the speed at times ·0015, ·0025 sec., and at a distance of 4 ft. from the breech. Estimate the muzzle velocity and compare it with the average velocity up to that point and with the velocity at half time; does it suggest that the acceleration was uniform?

113. Proof that the gradient of the tangent to the distance-time curve is the velocity of the body. Let PA represent the displacement of the body at the instant A, and QB the displacement at a later instant B. Produce the line PQ to any point R. Then the distance which the body moves during the time represented by AB (or PM) is represented by QM; hence the fraction $\frac{QM}{PM}\left(\text{or } \frac{RN}{PN}\right)$ represents the average velocity during the interval AB, when the force continues to act.

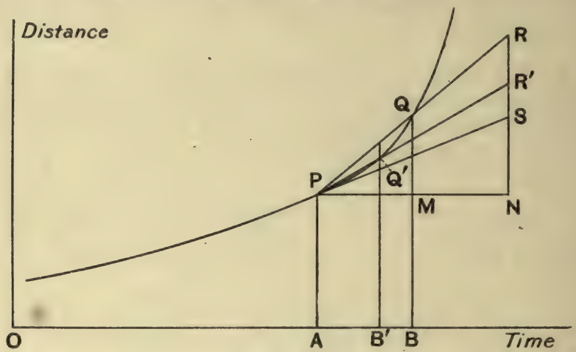


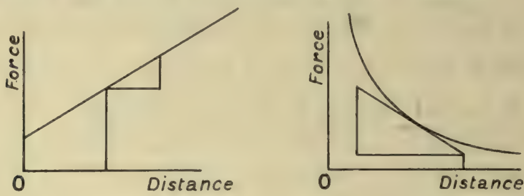
Fig. 81.

If we take Q nearer to P, at Q', then $\frac{R'N}{PN}$ represents the average velocity of the body after the instant A, when the force has continued to act on it for a shorter time AB'. If we take Q very close indeed to P, then the time during which the force continues to act is very short indeed, and the force will not have time to alter the velocity appreciably; so the average velocity during that interval is very nearly equal to the velocity with which the body would go on if the force had actually ceased to act at the instant A.

Now as Q approaches nearer to P, the line PQR turns round P, and comes continually nearer to some final position PS; (if it went further, it would cut the curve again on the other side of P). This position of the line is "the tangent at P." And as PQR approaches nearer to PS, the fraction $\frac{RN}{PN}$ becomes more nearly equal to the fraction $\frac{SN}{PN}$; hence $\frac{SN}{PN}$, or the gradient of the tangent, represents the velocity with which the body would go on if the force ceased to act on it at the instant A, i.e. to "the velocity at the instant A" (see Art. 9).

114. Rate of change of a quantity. We have hitherto treated the velocity of a body at an instant as being the distance which would be covered during the next unit of time if the velocity were unchanged. It may also be regarded as "the time-rate" at which the distance of the body from a fixed point changes; this word "rate," though we often use it in our everyday life, does not help us to understand the meaning of "velocity at an instant"; but a clear understanding of velocity at an instant (as above defined) helps us to understand the meaning of the phrase "rate of change" when applied to distances or other quantities. For instance, suppose that the force on a body changes with the position of the body, then the "rate per foot" at which the force changes with the distance of the body from a fixed point means the total change which would take place in the force when the body was displaced one foot if the rate of change of force with distance were to continue unaltered during this displacement. Thus if the force varies directly with the distance, as in a spiral spring (Fig. 82 (i)), the change of force for any displacement at all positions of the body is proportional to the displacement, so that the rate of change of the force with distance actually does remain constant. But if the force varies inversely with the distance (Fig. 82 (ii)), as in the case of the pressure of a definite quantity of compressed gas on a piston, then the rate of change of pressure with distance is not constant, nor is it the change in pressure for a unit displacement; it can be found by drawing a tangent to the force-distance curve at the point for which the rate of change is required, and measuring its gradient, just as was done with the distance-time curve.

In general, the gradient of the tangent gives the rate at which the quantity represented by the ordinate changes with a change in the quantity represented by the abscissa.



(i) (ii)

Fig. 82.

115. Acceleration as a rate of change of velocity.

Hitherto we have had to deal with cases where the rate of change of velocity per second (i.e. the acceleration) was constant, but if the force on the body is not constant, its acceleration will not be constant. We must extend the meaning of the word "acceleration" to suit such cases: it will now become "the acceleration at an instant is the change in velocity which would take place in the next unit time if the acceleration were to remain constant and of the same value as at that instant."

We can determine its value by drawing a tangent to the velocity-time curve at the corresponding instant, and calculating its gradient; but only moderate accuracy is attainable in this way.

Ex. 4. The following are observed velocities of a body, and corresponding times; plot the velocity-time diagram and calculate the acceleration at ·2, ·5 and ·7 sec. from the start, in ft. per sec. per sec.

t in secs.	0	·1	·2	·3	·4	·5	·6	·7	·8
v in ft. per sec.	0	·16	·31	·45	·59	·71	·81	·89	·95

Ex. 5. Use the velocity-time curve found in Ex. 2 to calculate the acceleration at ·05 and ·30 sec. from the start.

116. Instantaneous value of the acting force. When the resultant force on a body is not constant, the acceleration of the body will vary, but "the force at any instant" is measured by the product of the mass and the acceleration at that instant; for Newton's Second Law shows that if the force remained constant for unit time after that instant it would be measured by the product of the mass of the body and its acceleration, and under these circumstances the acceleration would remain at the same value as at the instant.

Therefore, when we have observed the distance of a body from a fixed point at a series of instants and have drawn a distance-time curve, we can deduce the value of the force acting on it at any time during the observed motion.

For instance, if the mass of the piston quoted in Exs. 2 and 5 is 30 lbs., we can find the resultant force on it at ·30 sec. from the beginning of the stroke. For the value of the acceleration at that instant is -44 ft. per sec. per sec.; hence the force is $30 \times (-44)$ poundals, or 1320 poundals or 41 lbs. wt. opposing the motion. This force is the resultant of the various forces acting on the piston, i.e. it is the difference between the steam pressure and the component along the piston-rod of the thrust along the connecting rod.

Ex. 6. Find the resultant force on the above piston at ·05 sec. from the beginning of the stroke.

Ex. 7. If the body in Ex. 4 weighs 4 lbs., find the force on it at ·2 sec. from the start.

117. Approximate method of deducing velocity from a distance-time curve. If we have observations of the distance of a body from a fixed point of its path at short equal intervals throughout its motion, and if these observations are accurate enough to give a distance-time curve that needs little or no smoothing, then we can deduce the velocity at a given instant by a method which though only approximate will probably give as accurate a result as by drawing a tangent to the curve. Take for example the observations given in Ex. 1, p. 201. The average speeds during successive twentieths of a second, calculated from these results, are 2·6, 5·8, 7·6, 8·0, etc. ft. per sec. ; if these values are compared with the values of the velocities at the mid-points of the corresponding intervals of time, as deduced from the velocity-time curve obtained by drawing tangents to the distance-time curve, they will be found to agree fairly well.

Hence we can often obtain a sufficiently accurate velocity-time curve by taking the average velocity during an interval as giving the velocity at the middle of the interval. This leads to the simple test for uniform acceleration suggested in Ex. 15, p. 29; if the average velocity during successive equal intervals of time increases by the same amount, the acceleration is uniform; when the acceleration is uniform, it is strictly true that the average velocity during an interval of any length is the velocity at the middle of the interval (Art. 15).

118. Calculation of distance by means of a velocity-time curve. If we know the velocity of a body at various instants of its motion, we can deduce its distance-time curve; this is the converse of the process described in Art. 112.

Consider first the case of a body moving with uniform velocity, v ft. per sec. say. If OA in Fig. 83 represents this velocity, then APQ is the velocity-time curve. Let OM, ON represent two periods, t_1, t_2 secs. say. The distance the body moves in the time $t_2 - t_1$ secs. is $v(t_2 - t_1)$ ft. Now PM represents v and MN represents $t_2 - t_1$, so the area $PMNQ$ represents their product, i.e. the distance run. As in the case of work and energy (which we have seen are represented by areas on a force-distance curve, see Arts. 55 and 70) we must be careful to determine the scale of areas, if we have to calculate the distance moved by computing the area on a velocity-time curve.

Ex. 8. On a velocity-time curve the scale is 1 in. to ·5 ft. per sec. and 1 in. to 3 secs.; what distance is represented by 7·5 sq. ins.?

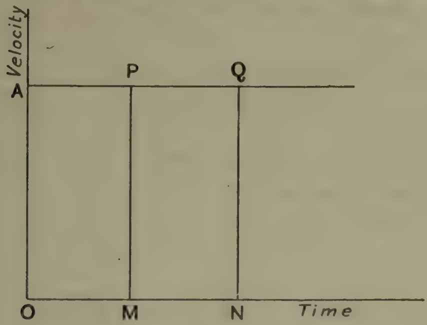


Fig. 83.

Next take the general case. The following proof follows the same lines as that in Art. 57, and will therefore be given more briefly.

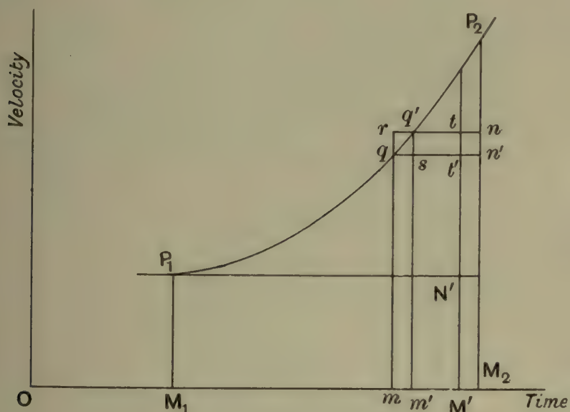


Fig. 84.

Let P_1M_1, P_2M_2 represent the initial and final velocities; subdivide M_1M_2 into a large number of equal parts, of which

mm', $M'M_2$ are two. If during the interval mm', the velocity had remained constant and equal to qm, the distance moved in that interval would have been represented by the area $qmm's$. If during the same interval the velocity had remained constant and equal to $q'm'$, the distance moved would have been represented by the area $rmm'q'$. As the velocity changes gradually from one value to the other, the distance must be intermediate between these values, so the error in taking it as represented by the area between the curve, the two ordinates and the time-axis must be less than the area $rqsq'$ or $tt'n'n$. Hence *the distance moved in the interval* M_1M_2 *is represented by the area between the curve* P_1P_2, *the two ordinates and the time-axis*, within a possible error represented by the area P_2N'. This possible error may be reduced to any extent we please by increasing the number of equal parts into which we subdivide the interval M_1M_2.

As an illustration of this process, which is much more accurate and practical than that of drawing tangents to a curve, consider the case of an engine and train, of total mass 100 tons, which at a certain instant have a speed of 12 miles an hour (or 17·6 ft. per sec.). Suppose that the engine is exerting a constant power, of 400 H.P., and neglect the frictional resistances to motion, which do not become considerable until high speeds are reached. In each second the kinetic energy is increased by $400 \times \frac{33000}{60}$ ft. lbs. or $220{,}000 \times g$ ft.-poundals. Initially it is $\frac{100 \times 2240}{2} \times 17{\cdot}6^2$ ft.-poundals; so at any time t secs. later it is

$$\frac{100 \times 2240}{2} \times 17{\cdot}6^2 + 220{,}000gt \text{ ft.-poundals.}$$

If its velocity is then v ft. per sec., its K.E. is $\frac{100 \times 2240}{2} v^2$ ft.-poundals; hence $\frac{100 \times 2240}{2} v^2 = \frac{100 \times 2240}{2} 17{\cdot}6^2 + 220{,}000gt$,

$$\text{or } v^2 = 17{\cdot}6^2 + 63{\cdot}25 \times t.$$

So we can calculate v corresponding to any value of t, and draw the velocity-time curve. For example, for 4, 10, 16, 20 and 30 secs. the velocities are respectively 23·73, 30·69, 36·35, 39·68 and 46·97 ft. per sec. The velocity-time curve should be drawn to scales of 1 in. to 10 ft. per sec., and 1 in. to 5 secs. Then 1 sq. in. represents a distance of 50 ft. along the railway line, and a square of ·1 inch side represents ·5 ft. (The curve shows at once that the acceleration is not constant.) Counting up these little squares under the curve, we find that the distance run in the first 5 secs. is 108 ft., in 10 secs. is 247, in 15 secs. is 413 ft., in 20 secs. is 601 ft., in 25 secs. is 809 ft., and so on.

Ex. 9. Find in this way the distance run in 40 secs.

119. Application to uniform acceleration. A very useful application of the general theorem proved in the last Article is found in the case of constant forces, where the acceleration is constant, i.e. the velocity-time curve is a straight line.

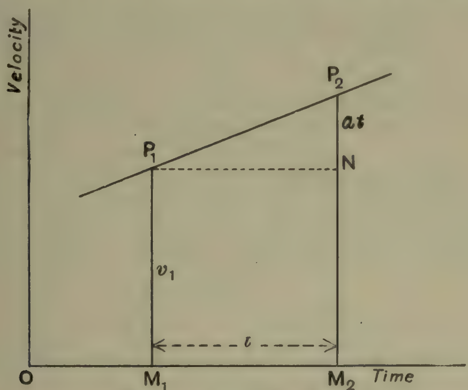


Fig. 85.

The value of the area which represents the distance moved can in this case easily be calculated, and we will use it to establish a general formula connecting distance, time and acceleration.

Suppose that the body has an initial velocity v_1 (ft. per sec.) at the beginning of the interval t (secs.) which we are considering; and suppose that the acceleration is a (ft. per sec. per sec.). This means that in one unit of time (1 sec.) the velocity will increase by a (ft. per sec.), so in time t (secs.) it will increase by at (ft. per sec.); hence at the end of the interval the velocity is $v_1 + at$ (ft. per sec.). If we call this final velocity v_2 (ft. per sec.), then

$$\mathbf{v_2 = v_1 + at} \quad \text{......................(i).}$$

This general formula depends only on the meaning of the words "constant acceleration."

In Fig. 85, let P_1M_1, P_2M_2 represent the initial and final velocities, then the area $P_1M_1M_2P_2$ represents the distance (s ft., say) moved during the interval t (secs.). We are, then, reckoning time, t, from the instant represented by M_1, and distance, s, from the position of the body at that instant. To calculate the value of this area, draw P_1N parallel to M_1M_2. Then P_1M_1 represents v_1, M_1M_2 represents t, and P_2N represents at; hence area $P_1M_1M_2N$ represents v_1t (ft.) and the area P_1NP_2 (or $\frac{1}{2} \times P_2N \times P_1N$) represents $\frac{1}{2} \times at \times t$ or $\frac{1}{2}at^2$ (ft.).

Hence $$\mathbf{s = v_1t + \tfrac{1}{2}at^2} \quad \text{......................(ii).}$$

We can put equation (ii) in another form; since P_1M_1, or NM_2, represents v_1, and P_2M_2 represents v_2, then P_2N represents $v_2 - v_1$, and the area of the triangle P_1NP_2 is $\frac{1}{2}(v_2 - v_1)t$.

Hence $$s = v_1t + \tfrac{1}{2}(v_2 - v_1)t = \frac{v_1 + v_2}{2}t \quad \text{...........(iii).}$$

It will be seen that the equation (iii) can be derived from (ii), and is not an independent result, as it appeared to be in Art. 15. The two were established by separate experiments to avoid introducing the above proof at that stage.

120. General formulae for uniformly accelerated motion. The equations (i) and (ii) of last Article are quite general; that is, if the proper signs are given to the quantities represented by v_1, v_2, a, s and t, they hold whether the final velocity is in the same or in the opposite direction to the initial velocity, whether we consider an instant before or after the time from which t is reckoned, etc. But it must be remembered that s does not necessarily represent the distance travelled by the body, but in all cases the distance it is from the point from which we are reckoning. (To make this point clear, Fig. 85 should be drawn for the case in which P_1 and P_2 are on opposite sides of OM.) From these two equations we can obtain a third, which is sometimes useful, by eliminating t from them.

From (i) we have $t = \frac{v_2 - v_1}{a}$; substituting this value of t in (ii) we get $s = v_1 \frac{v_2 - v_1}{a} + \frac{1}{2}a \frac{(v_2 - v_1)^2}{a^2}$, or $2as = 2v_1(v_2 - v_1) + (v_2 - v_1)^2$

or $$\mathbf{v_2^2 = v_1^2 + 2as} \quad \text{.....................(iv).}$$

This equation, like the others, holds whatever the sign of v_1, v_2, a or s.

Ex. 10. A train with the brakes on (to such an extent as to produce constant retardation) is observed at a certain instant to be travelling at 15 ft. per sec., and to come to rest after running 300 ft. further. How long before the instant of observation were the brakes applied, the train then running at 88 ft. per sec.?

Count time from the first observation, distance from the then position of the train. Then $v_1 = 15$ ft. per sec., $v_2 = 0$, $s = 300$ ft. Hence from (iv) $0 = 15^2 + 2a \times 300$, and $a = -\frac{3}{8}$ ft. per sec. per sec.

In order to apply equation (i), we must take v_1, the velocity at the beginning of the interval, to be 15 ft. per sec., and v_2, the velocity t secs. later, to be 88 ft. per sec. and $a = -\frac{3}{8}$ ft. per sec. per sec. as above; then $88 = 15 + (-\frac{3}{8}) \times t$, whence $t = -\frac{73 \times 8}{3}$ secs., or 3 mins. $14\frac{2}{3}$ secs. earlier.

Ex. 11. A ball is thrown vertically upwards with a velocity of 60 ft. per sec.; neglecting air resistance, find its height after 3 secs.

Ex. 12. Plot the distance-time curve for Ex. 11, on a scale of 1 in. to ·5 sec., and 1 in. to 10 ft.

121. Uniform motion in a circle. Consider the case of a body moving along a circular path with constant speed, such as a motor car running round a circular track, or a stone whirled round in a sling, or a bit of paint on the rim of a flywheel. Although its speed does not change, there must be some force continually acting on it, or it would move in a straight line instead of a circle; we have now to find what that force is.

Let us first imagine the body to move with velocity v (ft. per sec.) along the side AB of a regular polygon of N sides which is inscribed in the given circle, whose radius we will call R (ft.). When it reaches B, it would continue along the straight line

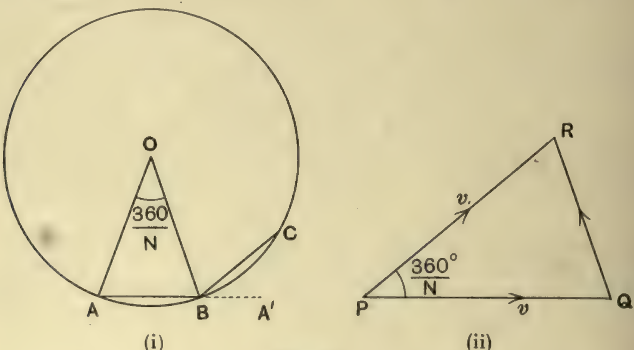


Fig. 86.

BA', but imagine that a sudden blow then changes its direction of motion to the next side of the polygon, BC, but without changing the speed of the body. Draw PQ parallel to AB to represent the velocity along AB, and PR parallel to BC to represent the velocity along BC; then QR represents the change of velocity which occurred at B. Now the angle AOB is $\frac{1}{N}$th of four right angles, and so also is $A'BC$; hence the angle $QPR =$ the angle AOB. Also, both the triangles AOB, QPR are isosceles triangles, hence they are similar. Hence the angle $PQR =$ angle

OAB = angle OBA; or QR is parallel to BO. The blow at B must therefore by Newton's Second Law act towards the centre O. Also

$$\frac{QR}{AB} = \frac{PQ}{OA} \quad \text{.........................(i).}$$

Denote the time in which the body traverses AB by t (secs.), then $AB = vt$ (ft.). So we can write (i) in the form

$$\frac{\text{change of vel. at } B}{vt} = \frac{v}{R}$$

or

$$\frac{\text{change of vel. at } B}{t} = \frac{v^2}{R} \quad \text{..................(ii).}$$

If now we suppose the number N to be indefinitely increased, the body will move with uniform speed in the circle, and the sudden blows merge into a continuous force directed towards the centre of the circle.

The left-hand side of equation (ii) now becomes the time-rate at which the velocity changes, i.e. the acceleration.

Hence when a body moves with uniform speed v (ft. per sec.) in a circle of radius R (ft.), its acceleration is $\frac{v^2}{R}$ (ft. per sec. per sec.), directed towards the centre; if m (lbs.) denote its mass, the force necessary to keep it moving in the circle is $\frac{mv^2}{R}$ (poundals) directed towards the centre of the circle.

122. "Centrifugal Force." This, then, is an instance of a force which varies in direction but not in magnitude. Since action and reaction are equal and opposite, the body may be considered to exert an opposing force, in consequence of its inertia, directed outwards from the centre and acting on the body which constrains it to move in the circle; this is called in everyday language "centrifugal force."

It is a somewhat misleading phrase, for it suggests that if this force prevailed the body would move outwards along the line of action of the force, i.e. along the radius. But if the external force which keeps the body moving in the circle suddenly ceases to act, the body will have nothing to oppose, and the "centrifugal force" will also cease to act; the body will then move along a tangent to the circle, not along a radius.

123. Illustrations of motion in a circle. As an instance of "centrifugal force," consider the behaviour of a weight hung by a string from the roof of a railway carriage which is travelling at a steady 30 miles an hour (or 44 ft. per sec.) along a circular track of radius 300 ft.

Fig. 87.

Fig. 87 (i) is a bird's-eye view of the weight, and its path is shown dotted. This is the "acceleration-diagram" (cf. Art. 79) of the weight; it has an acceleration of $\frac{44^2}{300}$ ft. per sec. per sec. directed towards the centre of its circular path. As we assume that the motion has been going on long enough for the weight and string to settle down into a steady position in the carriage, it has no other acceleration.

Fig. 87 (ii) is a cross-section of the carriage, looking in the direction in which the carriage is travelling; the centre of the circular track is to the spectator's left. This is the "force-diagram" of the weight; the only forces acting on it are the attraction of the earth (M poundals) and the tension along the string (T poundals). Call the inclination of the string to the vertical a. Then the resultant vertical force on the weight is $T\cos a - Mg$ poundals, and the resultant horizontal force is $T\sin a$ poundals, directed towards the left if the weight is displaced as shown in Fig. 87. Since there is no vertical acceleration, the former component vanishes, i.e. $T\cos a = Mg$. The horizontal component $T\sin a$ poundals produces a horizontal acceleration of $\frac{T\sin a}{M}$ ft. per sec. per sec.; we see by the acceleration diagram that the horizontal acceleration is $\frac{44^2}{300}$, i.e. $\frac{T\sin a}{M} = \frac{44^2}{300}$. From these two equations we get, by eliminating T and M, $\tan a = \frac{44^2}{330 \times g}$. Hence we find that a is about **11° 20′**.

So to a spectator sitting in the carriage there would appear to be a horizontal force, directed away from the centre of the circular track, acting on the weight, which kept the string inclined outward at a steady angle of 11° 20′ to the vertical; he would call this a "centrifugal force." But the actual forces acting on the weight are really only the attraction of the earth acting vertically downwards, and the tension of the string acting upwards and inwards towards the centre of the circular track; these two have an "unbalanced" horizontal component acting inwards, and so producing the inward acceleration needed to make the body move in a circle. There is no outward centrifugal force *on the body* at all, though to a spectator moving with the body there may appear to be one; but the body may be said to exert a "centrifugal force" on the carriage roof.

124. Take as another instance, a man skating at 15 miles an hour on the outside edge in a circle of radius 40 ft. Fig. 88 (i) gives his acceleration diagram as before, Fig. 88 (ii) the force-diagram. We know that he has to lean inward; suppose he does so at an angle of a to the vertical. Then the forces acting on him are (a) his weight (Mg poundals) acting vertically downwards through his centre of gravity, (b) some force exerted by the ice on his skate; resolve this into vertical (R poundals) and horizontal (S poundals) components. Since his foot is not "carried from under him" either to right or left by the reaction of the ice on his skate, this reaction must pass through his centre of gravity, since if it did not it would make him rotate about that point. Hence by the triangle of forces R and S are connected by the equation $\frac{S}{R} = \tan \alpha$. Further, since he has no vertical acceleration, $Mg = R$. The horizontal force on him is S poundals, which gives him a horizontal acceleration of $\frac{S}{M}$ ft. per sec. per sec.; hence from the acceleration diagram $\frac{S}{M} = \frac{22^2}{40}$. From these three equations we get $\tan \alpha = \frac{22^2}{40g}$. Hence $\alpha = 20° 36'$.

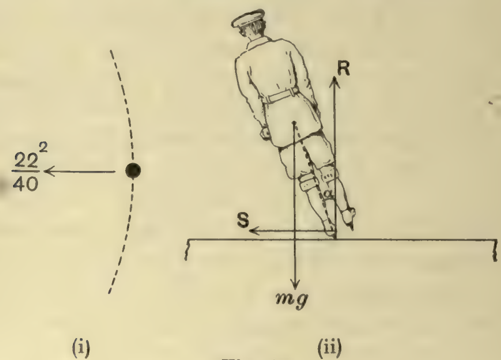


Fig. 88.

We can look at this case in another and better way. If the man is standing still and upright on one skate, there are forces acting on each particle of him; its weight (mg poundals) vertically downward, and the reactions of neighbouring particles. These forces on each particle are in equilibrium; hence the reactions of neighbouring particles compound into a single force of mg poundals vertically upwards. These reactions for all particles of the body compound into a single force (Mg poundals) acting through his centre of gravity, along the line joining his skate to his centre of gravity.

But when the man is moving in a circle, the forces on each particle are not in equilibrium; there must be a resultant force which produces in the particle a horizontal acceleration of $\frac{22^2}{40}$ ft. per sec. per sec. so this resultant force is $m\frac{22^2}{40}$ poundals. The weight of the particle is still mg poundals vertically downward; hence we can find by the triangle of forces what must now be the reaction of the neighbouring particles (BA in Fig. 89). This triangle is similar for every particle, hence the angle BAC is the same for all; call it a. Then the magnitude of the reaction of neighbouring particles on any particle of mass m pounds is $\frac{mg}{\cos a}$ poundals, and the direction of the force is the same for every particle. Hence this system of forces is similar to that in which an extended body is acted on by gravity, only the acceleration of "gravity" is increased from g to $\frac{g}{\cos a}$ and inclined to the vertical at an angle a; so these reactions will compound into a single force passing through his centre of gravity and, in order not to fall over, he must incline his body so that the line between his skate and centre of gravity is inclined to the vertical at an angle a.

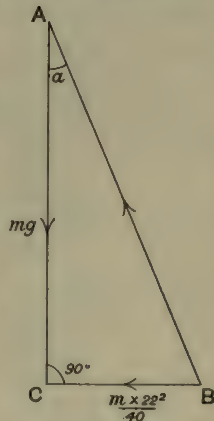


Fig. 89.

From Fig. 89, we see that $\tan a = \frac{22^2}{40g}$, as before.

For the same reason, the resultant forces on each particle, $\frac{m \times 22^2}{40}$, towards the centre of the circular path, compound into a single resultant force through the centre of gravity, of magnitude $M\frac{22^2}{40}$ poundals, where M lbs. is the mass of the whole body; for all these forces are parallel and produce the same acceleration in each particle, as in the case of gravity.

Hence we can treat the problem as if the whole mass was concentrated at the centre of gravity. Strictly speaking, the radius of the path is not the same for all the particles; but the method gives results which are very nearly correct if the radius is large compared with the size of the body.

Ex. 13. A man rides a bicycle at 20 miles an hour round a circular path of radius 30 ft.; calculate the angle at which he must lean inwards.

Ex. 14. Find the force of friction which is called into play sideways between the tyre and the ground in Ex. 13, if the man and bicycle weigh 200 lbs.

Ex. 15. If the path is "banked" to suit this speed of 20 miles an hour (i.e. sloped at right angles to its length so that it is at right angles to the bicycle) what is its gradient?

Ex. 16. What is the gradient across a circular track of radius ¼ mile banked for motor racing at 110 miles an hour?

Ex. 17. Find the additional weight which the bicycle has to support in Ex. 13, over the amount it supports when travelling in a straight line, if the man weighs 12 stone.

125. Curves in railway lines. In order to prevent the danger of trains being derailed when rounding curves, the lines are "banked," as is shown to be necessary in the last set of examples, by raising the outer rail to an extent depending on the average speed of the trains on the curve. If this speed be v ft. per sec., and the radius of the curve be r ft., the line joining the upper surfaces of opposite rails must be inclined to the horizontal at an angle α where $\tan\alpha = \frac{v^2}{rg}$. As the distance between the rails is 4 ft. 9 in., we can deduce the height by which the outer rail must be raised above the inner one.

Ex. 18. Calculate the super-elevation of the outer rail in inches for a speed of 60 miles an hour for a curve whose radius is 1500 ft.

Ex. 19. A coal truck weighing 15 tons, running on 4 wheels, has its centre of gravity 3 ft. above the rails. It runs at 15 miles an hour round a curve of radius 200 ft., the two rails being at the same level. Find the horizontal and vertical pressures of the rails on each wheel.

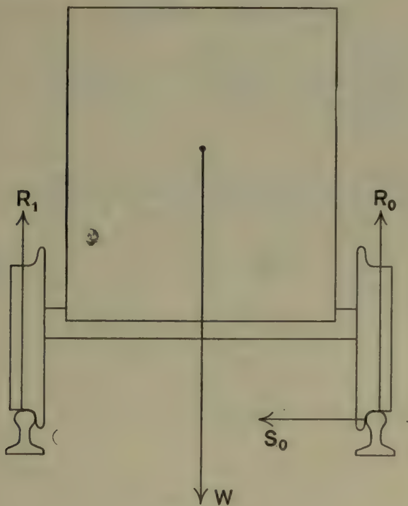


Fig. 90.

Use the notation shown in Fig. 90; the centre of the circular track is supposed to be to the left and the truck to be running from the spectator. There will be no pressure on the flange of the inner wheel. The forces R_1, R_0, S_0 and W must have a horizontal resultant acting through the C. G., of magnitude $15 \times 2240 \times \frac{22^2}{200}$ poundals, in order to make the truck move in a circle. We will call this T poundals.

Hence, resolving vertically,

$$R_1 + R_0 - W = 0 \quad \text{.................................(i),}$$

resolving horizontally

$$S_0 = T \quad \text{...(ii)}$$

and taking moments about the top of the outer rail, the counter-clockwise moments of the forces acting on the truck are $W \times \frac{4{\cdot}75}{2} - R_1 \times 4{\cdot}75$ poundal-ft. The moment of a force T horizontally through the C.G., towards the left, is a counter-clockwise moment of magnitude $T \times 3$ poundal-ft. Hence, since T is the resultant of the acting forces

$$W \times \frac{4{\cdot}75}{2} - R_1 \times 4{\cdot}75 = T \times 3 \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Since $T = 15 \times 2240 \times \frac{22^2}{200}$, and $W = 33600g$, we can calculate R_1, R_0 and S_0; the results must be halved to get the pressures on each wheel. It will be seen that even without banking, the inner wheel still supports a large proportion of the weight so that there is a large margin of safety, before the truck tends to overturn.

126. Motion in a vertical circle. As a further instance of motion in a circle, suppose a weight, tied to a string

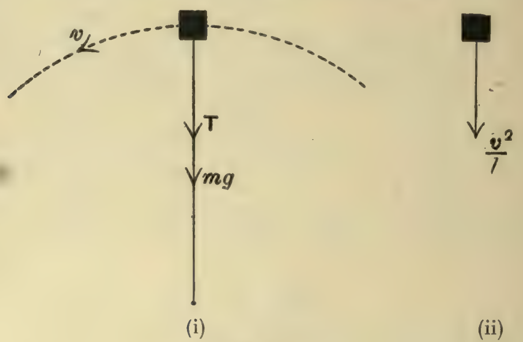


Fig. 91.

whose other end is fixed, is travelling round in a vertical circle. Its speed will not be uniform, for at different heights it will have different amounts of potential energy and therefore different amounts of kinetic energy; its speed will be least when it is passing its highest point. Suppose its speed is then v ft. per sec., and the length of the string is l ft., and the mass is m lbs. Call the tension of the string at this point T poundals. Then the forces on the body are T poundals and mg poundals, both acting vertically downwards. (In Fig. 91, (i) is the force diagram, and (ii) the acceleration diagram.) The resultant of these must produce an acceleration of $\frac{v^2}{l}$ ft. per sec. per sec. towards the centre of the circle, i.e. the resultant is a force of $\frac{mv^2}{l}$ poundals vertically downwards. Hence $T+mg=\frac{mv^2}{l}$.

We can determine the least speed of the body at the highest point at which it will keep on moving in its circular path, and not drop out of it, by finding the value of v from the above equation which makes $T=0$.

Ex. 20. Find the least velocity when $l=2$ ft.

Ex. 21. Determine the velocity at the lowest point in the case of Ex. 20. The decrease of potential energy is $mg\times 4$ ft.-poundals, so we can calculate the total kinetic energy.

127. Simple Harmonic Motion. We will now deal with a case in which a body moves under a force which is constant in direction but variable in magnitude; and we will introduce it by a mathematical device, not directly as in the case of a constant force.

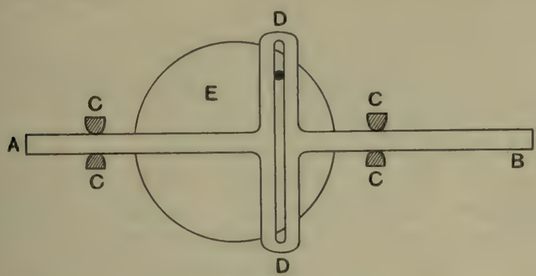


Fig. 92.

AB (Fig. 92) is a rod which is free to move in the direction of its length between guides *C*. To the rod is attached a cross-piece *D* with a slot at right angles to the length of the rod. A circular disc E with its centre opposite the centre line of the rod, carries a pin fixed to it, which fits without shake in the slot. If this disc is turned round on its centre, the pin causes AB to move backwards and forwards; and if the disc turns at a constant speed the rod is said to move with "Simple Harmonic Motion." The movement will be better understood if the apparatus is examined in action.

A more precise definition of simple harmonic motion (usually written S.H.M.) can be obtained geometrically.

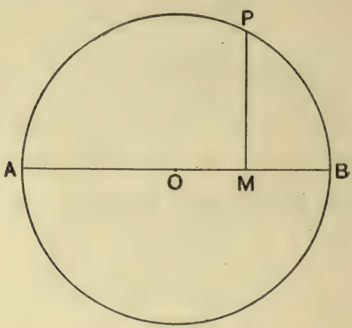


Fig. 93.

If a point P moves with constant speed round a circle, and a line PM is drawn perpendicular to a fixed diameter AB, then the point M moves with simple harmonic motion.

This is clearly the same arrangement as in the above apparatus. A little consideration of either will show that, starting from the end of its travel to the right, the point moves to the left, slowly at first and with increasing speed; the speed decreases on passing the centre of the circle, and vanishes at an equal distance on the other side. The point then returns through the centre, with similar changes of speed, to its starting point; it then repeats the cycle.

If v ft. per sec. is the speed of P, and r ft. is the radius of the circle, then P always has an acceleration of $\frac{v^2}{r}$ ft. per sec. per sec. directed towards O the centre of the circle. Draw a line po to represent this acceleration and resolve it into components pm and mo perpendicular and parallel to the fixed diameter AB. Then the acceleration represented by mo has a magnitude of $\frac{mo}{po} \times \frac{v^2}{r}$. But the triangle pmo is similar to PMO; hence

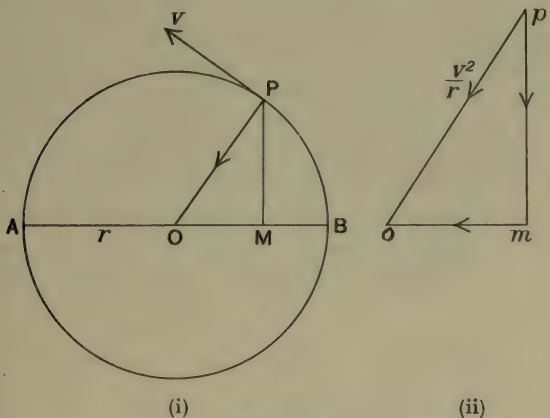


Fig. 94.

$\frac{mo}{po} = \frac{MO}{PO}$. Hence the component acceleration of P parallel to AB is $\frac{MO}{PO} \times \frac{v^2}{r}$. In this expression everything but MO is a constant for all positions of P in its path round the circle; hence the component acceleration of P parallel to AB is proportional to the distance of M from O. But the total acceleration of M is equal to the component acceleration of P parallel to AB. Hence *a body moving with S. H. M. has an acceleration which is proportional to its distance from a fixed point, and always directed towards that point.*

Therefore, a body moves with S. H. M. if the force on it is always directed towards a fixed point and is proportional to the distance of the body from that point.

It will be noticed that this was the case with the spiral spring supporting a weight, discussed in Art. 70; it is generally true of bodies moving under elastic forces, such as stretched strings, tuning forks, etc.; as these give out musical notes when they vibrate with sufficient frequency, the motion is called Harmonic. In many cases in everyday life the force on a body is more or less accurately proportional to the displacement of the body from its position of rest; e.g. a rolling or pitching ship, a pendulum, the surface of a rough sea, the balance wheel of a watch controlled by the hair spring; so in these cases the bodies oscillate with S. H. M. when displaced from their position of rest. Another case of a body which moves approximately with S. H. M. is the piston of a reciprocating steam engine; for the flywheel turns with nearly constant speed, and the connecting rod takes the place, though by no means accurately, of the slot and bar shown in Fig. 92.

128. We will deal here with one only of the very many important properties of this motion.

Suppose the body has a mass m (lbs.) and that its total travel AOB is $2r$ (ft.). Then the force acting on the body at the point M is as we have shown $m \times \frac{MO}{PO} \times \frac{v^2}{r}$ (poundals) where v (ft. per sec.) is the speed of the tracing point P in the circle. If we call MO, x ft., we can write the expression for the force $m \times \frac{v^2}{r^2} \times x$ (poundals).

Suppose T (secs.) is the time taken by the body to do one "complete oscillation," i.e. to move from B to A and back to B; we call T the "periodic time" of the oscillation (see Art. 4). This is therefore the time taken by P to go once round the circle. Hence we have $2\pi r = vT$.

Therefore the expression for the force becomes $m \times \frac{4\pi^2}{T^2} \times x$ (poundals).

If then in any particular case we are given the ratio between the force and the displacement, we can immediately determine the periodic time of the oscillation; for this ratio equals $m \times \frac{4\pi^2}{T^2}$. For example, in the case given on page 124 the ratio is $\frac{mg}{a}$; so the value of T in this case is $2\pi\sqrt{\frac{a}{g}}$ secs. It is important to notice that the value of the periodic time does not depend on r, the distance the body is displaced before it is released; in other words, provided the relation between force and displacement is constant, the periodic time is the same for a large as a small oscillation. This justifies the assertions made in Art. 4.

Ex. 22. The force on a body, of mass 3 lbs., is known to vary as the distance it is displaced from its equilibrium position. It is found that a force of 2 lbs. wt. will hold it at a distance of 18 ins. from this position. Find its periodic time when released.

Miscellaneous Exercises.

Ex. 1. Draw the graph of $s=3t+5t^2$, where s is distance moved by a body in ft. and t is time in seconds; hence determine the velocity at ·5 sec. from the start, and at the instant when the body has moved 6 ft.

Ex. 2. Repeat Ex. 1, the connection between s and t being given by the equation $s=3+5t^3$.

Ex. 3. A 12 lb. mass is hung from the end of a spiral spring. A force of 2 lbs. wt. extends the spring by ·3 in. Find the time of vibration of the 12 lb. mass.

Ex. 4. A skater describes a circle of 50 ft. radius at a speed of 20 ft. per sec. At what angle must he lean inwards?

Ex. 5. A motor car is running over a hump-backed bridge, whose upper surface is part of a circle of radius 50 ft. Find the speed of the car at which the wheels just lift off the road at the top of the arch.

Ex. 6. A string 4 ft. long has one end attached to a fixed point and carries a weight at the other. The weight is moving round in a horizontal circle such that the string has a constant inclination of 20° to the vertical. Determine the speed at which the weight moves.

Ex. 7. If the coefficient of friction of tyres on a greasy road is ·2, what is the greatest speed at which a corner (radius 50 ft.) can be taken just to avoid skidding?

Ex. 8. Taking the earth as a sphere of 4000 miles radius, calculate the reduction in weight of a pound at the equator in consequence of the earth's rotation.

Ex. 9. A train of mass 125 tons is started from rest by an engine of mass 100 tons. At the end of 0, 4, 8, 12, etc. secs. from the start the speed is observed to be 0, 1·4, 3·6, 6·3, 9·4, 12·0 and 13 ft. per sec. respectively. Draw the velocity-time graph, and from it deduce the acceleration of the train 16 secs. after the start. Hence find the tractive force on the train at this instant.

Ex. 10. From the velocity-time curve of Ex. 9, find by the method of Art. 118 the distance run in the first 16 secs.

Ex. 11. A moving body of mass 2 lbs. is brought to rest, the speeds at the beginning of successive seconds being 50, 34·3, 23·6, 15·5, 9·4, 4·9 and 1·6 ft. per sec. Draw the velocity-time curve, and find approximately the time spent in coming to rest and the retarding force at the end of the 4th second.

Ex. 12. From the velocity-time curve of Ex. 11, find by the method of Art. 118 the distance run by the body in coming to rest.

Ex. 13. A stone of mass ·5 lb. is whirled round in a vertical circle of radius 18 ins., so that its speed at the highest point is 10 ft. per sec. Find the tension of the string. If no energy is lost or gained in traversing the circle, find the velocity at the lowest point and the corresponding tension of the string.

Ex. 14. A truck of mass 4 tons moves at the rate of 60 miles an hour on a curve of 700 ft. radius; if the outer rail is not super-elevated, find the lateral pressure between the flanges of the wheels and the outer rail.

Ex. 15. A body of mass 200 lbs. is acted on by a resultant force which varies with the distance of the body from a certain point A; the force at a distance 0, 1, 2, 3, etc. ft. from A being 25, 26, 26, 24·8, 22·7, 20, 16·4, 11·6, 7, 3, 0 lbs. wt. Draw the force-distance curve (1 in. = 2 ft., 1 in. = 5 lbs. wt.)

and find the work done on the body while it moves 7 ft. from the point, in the same direction as the resultant force acting on it.

If the body passed A at a speed of 4 ft. per sec., calculate its velocity when it has moved 7 ft. from A.

Ex. 16. An electric tram-car weighs 6 tons. In starting from rest, the motor exerts a force which at the beginning of successive 5 secs. has the values 600, 580, 520, 420, 250, 150, 150 lbs. wt. Plot the force-time curve (scales 1 in. = 5 secs., 1 in. = 100 lbs. wt.). The car was found to have constant speed after 25 secs. from starting. Find the retarding force of friction at this speed. Assuming that this frictional resistance has remained constant throughout, determine by means of the area under the curve the momentum of the car after 25 secs. from starting, and hence its speed.

Ex. 17. Steam is shut off a train running at 60 miles an hour on the level, and the velocity at any time, t secs., later is given by the expression $47 \tan\left(\frac{724-t}{11\cdot 7}\right)^\circ$ ft. per sec. Plot the velocity-time curve (scales 1 in. = 100 secs., 1 in. = 10 ft. per sec.). By determining areas under the curve, find the distances run in 100, and 600 secs., and when the speed has dropped to 30 miles an hour and to zero.

Ex. 18. From the velocity-time curve of Ex. 17, find the acceleration at times $t=100$ and $t=600$ secs. Hence find the values of the frictional resistances, in lbs. wt. per ton, opposing the motion of the train at those instants. Hence determine the value of the constants a and b in the expression $a+bV^2$ lbs. wt. per ton for the frictional resistance, V being expressed in miles an hour. Verify the suitability of this form of expression by calculating from it the acceleration at 30 miles an hour and checking the result against the acceleration at the corresponding point of the velocity-time curve.

Ex. 19. H.M.S. "Drake," whose mass is 14000 tons, at a certain instant is moving at a speed of 3 ft. per sec., and her propellers are exerting 6200 H.P., which remains constant. Determine as in Art. 118 her speed and distance run by the end of 10, 20 and 30 secs., neglecting frictional resistances. (Take as scales for velocity-time curve 1 in. = 5 secs. and 1 in. = 2 ft. per sec.)

Ex. 20. From the velocity-time curve of Ex. 19 determine the acceleration at times 4 secs. and 25 secs. and hence deduce the propeller thrusts at those times.

APPENDIX ON APPARATUS

There is no need to describe the trolley and plane which is used for the majority of the experiments, as it is listed by many makers of scientific instruments in different designs of varying degrees of merit. Although less skill is required to manipulate one of the more highly finished sets with metal bed, etc., excellent results can be achieved with a very simple form if it is furnished with a satisfactory trigger to release the trolley as the vibrating spring makes its first passage across the centre line.

In order to assist the acquisition of concrete ideas of velocity and acceleration at an early stage it is convenient to fit a speedometer to the trolley, for a few preliminary experiments. This can be done by using a hand tachometer fitted with a rubber disc for measuring linear speeds; it can be fixed, by a bracket on the end of the trolley, in a vertical position so that the disc runs on the vertical edge of the plane, held against it by a spring from the trolley. If the scale of the tachometer is uniform, the needle will be seen to move with uniform angular velocity when the acceleration is constant; and the final velocity can be read after the applied force has ceased to act, even if the tachometer is not dead-beat. But it is not easy to ensure that a constant slope of the plane will give a constant speed, and this slope has usually to be changed for different loads on the trolley; the effective inertia of the mechanism is also considerable compared with the mass of a trolley. It is, therefore, preferable to employ very large loads, and to use a separate machine which can be made satisfactorily by fixing a board on the axles of two pairs of perambulator wheels about 14 in. in diameter; the pupil sits on this board, and reads a speedometer driven by a thin cord from a pulley fixed on one of the wheels, like the belt pulley on a motor-bicycle. The machine is drawn forward by a string which passes under a pulley fixed to a wall and then over a pulley fixed to the ceiling, carrying a weight sufficient to overcome friction (about 2 lbs. wt.), and, say 4 ft. lower, another weight to produce acceleration (say 3 lbs.). Thin string and light pulleys are of course sufficient for these small forces. The string should be attached to a spring balance fixed to the machine, so that the boy can observe the tractive force and the frictional resistance (the latter in the interval between the two weights reaching the floor). If the duration of runs from rest of various lengths is measured with a stop-watch, these times will be approximately in the same proportion as the final speeds attained and the acceleration corresponding to an observed tractive force can be calculated; the equations $F = ma$, $Fs = \frac{1}{2}mv^2$ can then be verified. There is no doubt that this experiment is more illuminating to the immature mind than a great deal of manipulation of delicate apparatus, in which the forces called into play seem to him insignificant.

Of the same order is an experiment to illustrate the First Law, by laying four or five lead pencils on the floor, parallel to one another and a few inches apart, with a board on them; a boy who stands on this board and attemps a long jump at right angles to the pencils will realise the need of a horizontal reaction from the ground to set him in motion horizontally.

Another useful experiment illustrates the effect of reducing the radius of gyration of a rotating body, but this needs a special piece of apparatus consisting of a circular board a foot in diameter supported on a circular ring of bicycle balls near its circumference, which balls rest in a circular ball-race in a base-board. The boy stands upright on this turn-table with his arms stretched out horizontally and a 4 lb. weight in each hand, and is set to rotate slowly; if he then bends his arms so that the weights are brought towards his shoulders, his speed of rotation will increase, but can be reduced at will by increasing his radius of gyration again.

Although the apparatus described on page 158 is best adapted for explaining the principles involved, accurate experimental results can be produced with less care by using the following form. Take a piece of clock-spring about 12 in. long; bend it into a circle so that the ends overlap about half an inch; clamp these ends firmly on the vertical end face of a trolley, so that the spring stands out in a horizontal circle, forming a spring buffer. It will be found that the loss of energy in a collision with a trolley at rest is inappreciable; the only points in manipulation which demand care are the sloping of the plane for constant speed and the synchronising of the vibrating springs.

DESCRIPTIVE QUESTIONS.

1. Describe an experimental method by which the displacement of a trolley during one-eighth of a second can be determined, when you are provided with a watch which only records seconds.

2. Define Uniform Speed. How would you test whether (1) a motor-car, (2) a trolley, was moving with uniform speed?

3. A railway train is slowing down as it enters a station; explain fully what you mean by "its velocity five seconds before it stops."

4. You are given the distance-time diagram for a body's motion; how would you determine whether the speed was uniform during any period, and the value of that uniform speed?

5. Give instances from common life, other than those mentioned in the text, of the effect of inertia, e.g. carpet-beating.

6. State Newton's First Law of Motion. Why do you believe it to be true?

7. Define Force, and show how the definition follows from Newton's first law.

8. Describe an experiment to show that the method of measuring forces used in Statics, by the pull of a string carrying a weight, is not suitable in all cases in Dynamics; how far is it safe to use this method?

9. Describe fully an experiment to show that when a body moves from rest under a constant force, its velocity at any instant is proportional to the time elapsed from the start.

10. What do you mean by Uniform Acceleration? If a motor-car had a uniform acceleration, how would the needle of the speedometer move over the dial?

11. Describe fully an experiment which shows that if a body moves under a constant force, its average velocity during any interval is half the sum of its velocities at the beginning and end of the interval.

12. If you were given a tracing of the motion of a body, how would you determine from it whether the acceleration was uniform?

13. How can you prove that all bodies, whatever their weight, have the same uniform acceleration when falling freely?

14. How can you determine by direct experiment the acceleration of a body falling freely?

15. Describe a method of determining the periodic time of a short pendulum.

16. How do we know that a pendulum has the same periodic time whether its swing is short or very short?

17. Describe an experiment to show that a body has a uniform retardation when its motion is opposed by a constant force.

18. Prove that a body thrown vertically upwards will return to its starting point with the same speed as it started with, if the air resistance is neglected.

19. A trolley runs down an inclined plane; describe how you can determine without calculation the resultant force on the trolley along the plane, including the frictional resistance.

20. Describe an experiment to show that the acceleration of a body is proportional to the force acting on it.

21. Prove that the forces needed to produce the same acceleration in different bodies made of the same material are in the same proportion as the quantities of this material in the bodies.

22. How would you compare the quantities of salt in two pots of salt-and-water which were known to be of different strengths?

23. How are the respective quantities in consignments of the following materials determined: coal, petroleum, compressed hydrogen, treacle, gold, diamonds, sulphuric acid, radium?

24. How can you test whether the masses of two bodies made of different materials are equal?

25. How can you prove that the masses of two bodies are in the same proportion as their weights?

26. How does the mass of a body differ from its weight?

27. Can you say that the mass of a body is greater or less than its weight? If not, explain why it is impossible to compare these quantities.

28. Explain fully what is meant by the statement "the mass of this body is 6·5 lbs."

29. Define the kinetic unit of force.

30. Show that the value in poundals of the force acting on a body is equal to the product of its mass in pounds and its acceleration in ft. per sec. per sec.

31. How is the kinetic unit of force related to the unit of force generally used in statics? Explain how you get the relation between them.

32. What do you mean by resolving a displacement into two others in given directions?

33. Prove that if a body moves with uniform velocity relative to a second body which itself has a uniform velocity in another direction, the former actually moves with uniform velocity.

34. Prove that if a body has simultaneously two uniform velocities represented by two sides of a triangle taken in order, its actual velocity is represented by the third side.

35. What do you mean by the vertical component of the velocity of a projectile at a given instant?

36. A body alters its velocity, from one known value to another; how can you determine the change which has taken place in its velocity?

37. Describe an experiment to show that the acceleration of a body falling freely is not affected by any initial horizontal velocity.

38. Describe an experiment to show that the velocity of a body in any direction is not affected by a force acting on the body at right angles to that direction.

39. A stone is projected at a known elevation and with a known speed; describe how you can determine its position at any subsequent instant.

40. A stone is projected at a known elevation and with a known speed; how can you determine its velocity at any subsequent instant?

41. State Newton's Second Law of Motion.

42. How does Newton's second law justify us in calculating the motion of a body acted on by several constant forces as though only a single resultant force acted on it?

43. Explain how the fact that we always get the same result in an experiment with a trolley, whatever time of day we repeat it, shows that each force acting on a body produces its result independent of any other force that may be acting on the body.

44. When does a force do work? Does the spring of a watch do more or less work when it is in the pocket of a man walking uphill? Does a locomotive do more work when it goes a mile up an incline than when it goes the same distance down an incline? Does the engine of a steamer do more work when the steamer goes a mile up stream than when it goes a mile down stream? Give your reasons for each answer.

45. Obtain the relation between a ft.-lb. and a foot-poundal.

46. Show that the work done in pulling a body up a frictionless incline does not depend on the gradient, but only on the vertical height through which the body is raised.

47. Explain fully why less work is done in pulling than in pushing a roller across a soft lawn.

48. Explain the disadvantage of using a short rope when towing a boat from the bank.

49. State the laws relating to friction between dry solids, and describe methods by which they can be roughly verified.

50. What is the meaning of "coefficient of sliding friction"?

51. A solid body is sliding on a rough inclined plane; explain how the frictional resistance to motion can be calculated when the coefficient of friction is known.

52. Explain why it is advisable not to apply the brakes to a moving vehicle to such an extent as to lock the wheels.

53. Discuss the advantage of loading the driving wheels of a locomotive and of coupling the wheels.

54. Why are continuous brakes (i.e. brakes on every wheel throughout the train) fitted to passenger trains?

55. Explain why the tractive force needed for a carriage is reduced by having large wheels and small axles.

56. Define Power and Horse-power.

57. Explain why it is necessary to have a considerable tension in the "slack" side of a belt used to transmit power.

58. Describe the method of measuring the horse-power of a motor by a rope brake.

59. Prove that the work done on a body during any displacement is represented by the area bounded by the ordinates to the force-displacement curve through the points representing the initial and final positions, the curve and the horizontal axis.

60. What do you mean by an Indicator Diagram for a steam engine?

61. Describe the method of taking an indicator diagram, and of calculating from it the mean effective pressure.

62. What do you mean by the average value of a force which changes with the position of the body on which it acts?

63. Explain the term "mean effective pressure."

64. What do you mean by Indicated Horse-power; how does it depend on the dimensions of the cylinder, the mean effective pressure, etc.; why is its value greater than that of the brake horse-power?

65. Define Energy, and give examples.

66. State the law of conservation of energy. Give an example to illustrate it.

67. What is the meaning and value of Joule's equivalent? What do you mean by a British thermal unit?

68. Distinguish between Potential and Kinetic energy. Illustrate the difference by considering a bent bow fitted with an arrow ready to be shot vertically upwards, and the subsequent changes of energy.

69. Obtain by means of the Second Law of Motion an expression for the kinetic energy of a body of mass m moving at a speed v, and state the units involved.

70. Describe an experimental method of verifying the expression for the kinetic energy of a body.

71. A weight is hung by a spiral spring and oscillates vertically; show how the kinetic energy of the weight at any given distance from its equilibrium position can be determined by measuring an area on its force-displacement diagram.

72. Give instances of the storage of energy.

73. Explain fully why it is essential for smooth running to fit a flywheel in a motor car.

74. Define the radius of gyration of a flywheel.

75. Describe an experimental method of measuring the radius of gyration of a flywheel mounted in bearings whose friction we may neglect.

76. Describe a method of taking into account the effect of bearing friction in determining by experiment the radius of gyration of a flywheel.

77. State Newton's Third Law of Motion.

78. Assuming the truth of the Third Law of Motion, show that the stronger team in a tug-of-war can pull the weaker over the line without pulling the rope harder than the weaker team is doing. On what body is the winning team exerting a greater force than that exerted by the losers?

79. Describe an arrangement by which we can study the motion of two bodies which exert a force on one another, but whose motion is not affected by any forces other than their mutual action and reaction. What is found to be the most suitable way to measure the quantity of motion produced in each body by this force; why are velocity and kinetic energy unsuitable measures of the quantity of motion in problems of this kind?

80. Define the momentum of a body.

81. Show from the Second Law of Motion that the change of momentum in a body under the action of a constant force is measured by the product of the numbers of units in the acting force and the time of action.

82. Show from the Third Law of Motion and the result of Ex. 81 that there is the same change of momentum in each of two bodies which act on one another with constant force.

83. State the principle of Conservation of Momentum, and describe one experiment to verify it.

84. State the two most usual methods of determining the average value of a variable force acting on a body. Would you expect them to give the same result?

85. Give an explanation of the pressure exerted by the wind on the sails of a square-rigged ship running before the wind.

86. Why cannot an undershot water wheel extract all the kinetic energy from the stream of water?

87. Explain why a kite can be kept at rest in the air although both its weight and the string pull it downwards.

88. Explain, without calculations, why it is possible for a sailing boat to travel faster than the wind.

89. Describe the construction of a de Laval steam turbine.

90. Explain in general terms why a turbine can extract a greater proportion of the energy of the jet than can an undershot water wheel.

91. Describe an experiment to demonstrate the reaction on a pipe through which water flows, when the pipe causes a change in the direction of the momentum of the water.

92. If water flows through a straight pipe whose bore gets smaller, how does the velocity change? Does a pound of water possess greater or less momentum after passing the constriction? Whence comes the force needed to produce this change of momentum? Is the force of fluid pressure on the pipe from broader to narrower part, or the opposite way? If the pipe widens out again to its original bore at a later point, will there be any resultant force on the pipe? Will the fluid pressure recover its former value?

93. Describe one toy reaction-motor.

94. Describe the jet method of propulsion, as applied to H.M.S. "Waterwitch." Why was it not adopted for other ships?

95. Give an explanation of the action of a screw propeller.

96. Prove that the gradient of the tangent to a distance-time curve measures the velocity.

97. Prove that the gradient of the tangent to a velocity-time curve measures the acceleration.

98. Given the mass of a body, and its distance-time curve, explain how you can determine the value of the resultant force on it (1) at any instant, (2) at any distance.

99. Prove that if the average velocity of a body during successive equal intervals of time increases by the same amount, the acceleration is uniform.

100. Prove that in uniform acceleration the average velocity during any interval of time is the same as the velocity at the middle of that instant.

101. Prove that the change of distance of a body from a fixed point during any time-interval is measured by the area between the velocity-time curve, the ordinates corresponding to the beginning and end of the time-interval, and the time-axis.

102. What physical quantities are represented by an area on (1) a velocity-time curve, (2) a force-time curve, (3) an acceleration-time curve, (4) a force-distance curve?

103. Establish the equation $v_2 = v_1 + at$ for uniform acceleration.

104. Establish the equation $s = v_1 t + \frac{1}{2}at^2$ for uniform acceleration.

105. Establish the equation $v_2^2 = v_1^2 + 2as$ for uniform acceleration.

106. Prove that when a body moves in a circular path at constant speed, its acceleration is $\frac{v^2}{r}$ towards the centre.

107. A body moving in a circle is said to exert centrifugal force; what does this force act on?

108. Why are we justified in stating that the resultant of the forces acting on an extended body, which moves at constant speed in a circle, must pass through the centre of gravity of the body?

109. Define Simple Harmonic Motion.

110. A body moves with S.H.M.; what do you mean by its amplitude of motion and periodic time?

111. Show that the periodic time of a body moving with S.H.M. does not depend on the amplitude of the motion.

ANSWERS TO EXAMPLES

Chapter I. Examples.

3. Train 80·7, pigeon 80, ft. per sec. **4.** 20·45 m. p. h.; 21·2 m. p. h.; 19·7 m. p. h. **5.** 19·8 m. p. h.; yes. **6.** 26·86, 24, 28·125, 24·49 m. p. h.; 25·76 m. p. h.; yes; no. **7.** 14·1, 19·2 secs. **8.** 46·9, 32·4, 41·1 m. p. h.
9. 47, 23·5, 7, 14, 21·3 cm. per sec. **10.** 45 m. p. h.; 230 secs.; 27·0 m. p. h.

Chapter I. Miscellaneous Exercises.

1. 4·62, 3·75 ft. per sec. **2.** 26 miles. **3.** 4·09, 1·5 m. p. h. **4.** 25, 15, 18·75, 20 m. p. h. **5.** 3·33 ft. per sec. **6.** 3·43 m. p. h. **7.** 25·6 m. p. h. **8.** 25,133 mi.; 1047 m. p. h. **9.** 66,000 m. p. h. **10.** 1100 yds. **11.** 6 m. p. h. **12.** 2·27 °/₀. **13.** 4·54 secs. **14.** 4, 3·48 m. p. h. **15.** 130 revs. per min. **16.** 27. **17.** 8·86 knots; 10·2 m. p. h. **18.** 42·4 ft. per sec.; 115·8 m. p. h.

Chapter II. Examples.

3. 94·7 cm. per sec. per sec. **4.** 5 m. p. h.; 2 mins.
5. 8·48 cm. per sec. **9.** 160 ft. per sec.; 400 ft.
10. 8·48 cm. per sec.; 2·12 cm. per sec. per sec. **11.** 288 ft. per sec.; 896 ft.
12. 1·25 sec.; 23·75 cm. per sec.; 11·85 cm. per sec. **13.** 1·52, 4·56, 7·60, 10·64 cm. per vib. **14.** 1·85, 5·80, 9·45, 13·30, 17·10, 20·90 cm. per vib.
15. No. **16.** 2943, 1471·5 cm. per sec.; 4414·5 cm. **17.** 16·1, 64·4, 144·9, 257·6, 402·5 ft. **18.** ·273 sec. **20.** ·61 sec. **21.** 18·96 ft. per sec.
23. 2·045 secs. **24.** 146·7, 44 ft. **26.** 6·21, 6·21 secs.; 200 ft. per sec.
27. 6·453 ft. per sec. per sec.

Chapter II. Miscellaneous Exercises.

1. ·733 ft. per sec.; 1 mile. **2.** ·0267 sec. **3.** 55 yds. **4.** 2 mins.; 1·67 miles. **5.** 600 yds. **6.** 1·5 miles. **7.** 24560 miles per hr. per hr. **8.** 4·89 ft. per sec. per sec. **9.** 96·32 ft. per sec. **10.** 129·2 ft. **11.** 4143 ft. **13.** 62·5 ft.; after 12·5 secs. **14.** 3·525 secs. **15.** 32·2 ft. per sec. per sec. **16.** 780 ft. **17.** 1 cm. per sec. per sec.; 3·5 cm. per sec. **18.** 326 ft. **19.** 32 ft. per sec. per sec. **21.** 1 sec., 2 secs. **22.** Highest point 80·5 ft. at 2·23 secs. **23.** ·00167 sec. **24.** 1·1, 11·3 secs. **25.** 144 ft.; 4 secs.; 40 ft. per sec.; 64 ft. per sec. **26.** The latter is $\sqrt{2}$ times the former.

Chapter III. Examples.

1. 98·1 cm. per sec. per sec. **2.** 2·3 ft. per sec. per sec. **4.** 1120 lbs. wt.; 0; 560 lbs. wt.; 1085 lbs. wt. **5.** 168 lbs. wt.; 0; 84 lbs. wt.; 162·8 lbs. wt. **8.** 237,000 dynes. **9.** ·536 ft. per sec. per sec.; 32·1 ft. per sec. **10.** 4930 pdals; 153 lbs. wt. **11.** 32,120 pdals, 998 lbs. wt. **12.** 21,390 pdals, 664 lbs. wt. **13.** 5074 pdals, 157·5 lbs. wt.

Chapter III. Miscellaneous Exercises.

1. 400 pdals. **2.** 100,000 dynes. **3.** 5·36 ft. per sec. **4.** 1·36 lbs. **5.** 657,000 pdals, ·917 tons wt. **6.** 96 pdals. **7.** 48 pdals. **8.** 2240 lbs. wt.; 71,680 pdals; 358,400 pdals; 1·6 ft. per sec. per sec. **9.** ·96 ft. per sec. per sec. **10.** 61,600 lbs. wt.; 9·32 secs., 410·1 ft.; 10·8 secs., 474 ft. **11.** 6400 pdals, 200 lbs. wt.; 7000 pdals, 218·75 lbs. wt.; 5600 pdals, 175 lbs. wt. **12.** 336,000 pdals, 10,500 lbs. wt.; 112,000 pdals, 3500 lbs. wt. **13.** Former is 443 times the latter. **14.** 13,840. **15.** 31·4 m. p. h. **16.** 800,000 ft. per sec. per sec.; 25,000 pdals, 781 lbs. wt.

Chapter IV. Examples.

1. ·846 mi., S. 57° E. **2.** 42·7 ft. **3.** 2·828 mi. N. and 2·828 mi. W. **4.** 6·363 mi. N.; ·707 mi. E. **5.** 5·414 mi. **6.** 8·54 m. p. h.; ·534 mi. 330 yds.; 3·75 min.; yes. **8.** 4·05 min.; ·5 mi.; ·54 mi. **10.** 36° 15′. **11.** 15° 57′ to vertical. **13.** 32·19 ft. per sec. vertically.

14. 25 ft. per sec. at 53° to former road. **15.** 22 ft. per sec. at 60°. **18.** 81·52 ft. per sec. at 52° 11′ to horizontal. **19.** 58° 9′. **20.** 62,120 ft. **21.** 4160 ft. **22.** 45°; 60′. **23.** 40·5 ft. up, 56·6 ft. along; 24·8 ft. up, 169·7 ft. along. **24.** 86·7 ft. per sec. at depression of 64° 18′. **25.** 4 ft. per sec. to E.; 3 ft. per sec. to N.; yes.

Chapter IV. Miscellaneous Exercises.

2. 1230 ft. **3.** 16·7 ft. per sec.; 19·24 ft. per sec. at 29° 57′. **4.** 2·5 secs.; 3500 ft. **5.** 13·06 secs.; ·0969 sec.; 192·7 ft. **6.** 622 ft. **7.** ·964 sec.; 3·7 ft.; 3·7 ft. up, 9·2 ft. along; 18·4 ft. per sec. **8.** 26·46 ft. per sec.; S. 11° E. **9.** N. 21° 48′ W.; 10·77 m. p. h. **10.** 77·8 ft. **11.** 3 m. p. h.; 78·7 secs.; 68·2 secs.; 100 yds. **12.** 27·7 m. p. h. **13.** 7·3 ft. **14.** 110·8 ft. per sec. **15.** 834 yds. **17.** 2067, 752 ft. per sec.; 23·5 secs. **18.** 1129 ft. per sec. **19.** S. 87° 15′ W., 10·62 mi. per day. **20.** 3 mi. **21.** Vertically; 16 ft. per sec.; 1 sec.; 8·8 ft. **22.** 3° 15′ behind the line; no. **23.** 7·52 lbs. **24.** 241 ft. per sec.

Chapter V. Examples.

2. 1472 million ergs. **3.** 36·06 million ft.-poundals; 1,120,000 ft.-lbs. **5.** 8·73 lbs. wt. **6.** 3·39 lbs. wt. **7.** ·59 lb. wt. **8.** 17·46, 33·9, 5·9 ft.-lbs. **9.** 5 tons wt.; 32·2 ft. per sec. **12.** 5·28 ft.-lbs. **13.** 2960 ft.-lbs. **14.** 1·12 lbs. wt. **19.** 480 H.P. **20.** 778·7 H.P. **23.** ·3055 H.P. **24.** ·133 H.P. **26.** ·145 H.P. **30.** ·057 H.P. **31.** 1 sq. in. to 2000 ft.-lbs. **32.** 5940 ft.-lbs. **33.** 30·05 ft.-pdals. **34.** 9·046 sq. ins.; 5654 ft.-lbs. **35.** ·521 sq. in.; ·833 ft.-pdals. **37.** 32·14 H.P.

Chapter V. Miscellaneous Exercises.

1. 18 turns; 282·7 ft.; ·19 H.P.; ·0636 H.P. **2.** 754 ft. per min.; 1·805 H.P. **3.** 60 lbs. wt. **4.** 1509 ft. per min.; 2187 lbs. wt. **5.** ·303 H.P. **6.** 27·1 H.P. **7.** 27·4 I.H.P. **8.** 15·15 H.P. **9.** 3·7 sq. ins.; 60·8 lbs. per sq. in.; 1850 ft.-lbs.; 5·6 H.P. **10.** 11·67 lbs. wt.; 2·43 ft.-lbs. **11.** 495,000 ft.-lbs. **12.** ·155 H.P. **13.** ·25 lb. wt. **15.** 266 lbs. wt.; 94 lbs. wt.; ·354; ·75 H.P. **16.** 628 ft. per min.; 52·5 lbs. wt. **17.** ·318 H.P. **18.** 373, 557 H.P. **19.** 53·8 H.P. **20.** 15·45 H.P.

Chapter VI. Examples.

3. 216 ft.-pdals; 6·71 ft.-lbs.; 1·12 lbs. wt. **4.** 78,750 ergs; 5250 dynes. **6.** 6 secs. **7.** 70,300 ft.-pdals; 2184 ft.-lbs. **9.** 345 B. TH. U.; 243·5° F. **10.** 38·8 ft. **11.** 3·72 ft. per sec. **13.** ·8, ·44 lbs. wt. **14.** 2·1 ft. per sec. **16.** 8·23 lbs. wt. **17.** ·2936 ft.-pdals; 1·71 ft. per sec. **19.** 30 ft.-pdals; 2·93 ft. per sec. **20.** 13·16 ft.-pdals; 1·94 ft. per sec. **21.** 150·3, 37·6, 150·3 pdals. **22.** 104 ft. **23.** 34·36 ft. per sec.; 157 ft. **25.** 19 million ft.-pdals.; 175·6 tons wt. **27.** 3·54 million ft.-pdals. **28.** 10,530 ft.-pdals. **29.** 3·18 ins. **30.** 20·37 million ft.-pdals. **31.** 8270 ft.-pdals. **32.** 2 ft. 10 ins.

Chapter VI. Miscellaneous Exercises.

1. 1,150,000 ft.-lbs.; 198 million ft.-lbs.; 172 lbs. **2.** 798° F. **3.** 50·2 lbs. wt.; 40·1 ft. per sec. **4.** 1·37 H. P. **5.** 69·3° F. **7.** 2382 B. TH. U. **8.** 1403 lbs. wt. **9.** 11,750 lbs. wt. **10.** $1{\cdot}73 \times 10^8$ ft.-pdals; 57,600 tons wt. ("tons" should be "lbs." in the question). **11.** 201 tons wt.; 1560 ft. **12.** 2·31 m. p. h. **13.** 241 H. P. **14.** 5·71 B. TH. U.; 4460 ft.-lbs.; 780·7 ft.-lbs. **15.** 1132 ft. per sec., 336,400,000 ft.-pdals. **16.** 12·27 ft. per sec.; 140·4 lbs. wt. **17.** 18·32 ft.; 73·3 ft.-lbs.; 40 lbs. wt. **18.** 11,880 ft.-lbs. **19.** 4·1 ins. **20.** 33 ft. **21.** 109. **22.** 60,000, 350,000 ft.-pdals. **23.** 158 lbs. wt.; 9 ft. 2 in. **24.** 111,200 ft.-lbs.; 506 lbs. wt. **25.** 3 ft. 1·7 ins. **26.** 29,100, 34,900 ft.-pdals. **27.** 2 : 1. **28.** 10·84 million ft.-pdals; 280·5 lbs. wt. **29.** 1·27 million ft.-pdals; 207·8 r. p. m. **30.** ·019, 50, ft.-lbs., 52·9 H. P. **31.** 1·74 ft. **32.** 240, 3603 ft.-pdals; 3·65 ins.; 1·74 secs. **33.** ·96 H. P.; 386 lbs. wt.

Chapter VII. Examples.

1. 89·2 cm. per sec. per sec.; 53,500 dynes. **2** and **3.** ·785 ft. per sec. per sec.; 660 pdals. **4.** 2·33 ft. per sec. per sec. **5.** 9·65 ft. per sec. **6.** 39·24 cm. per sec. per sec.; 612,000 dynes. **8.** ·975 ft. per sec. per sec. up plane. **9.** 2·15 ft. per sec. per sec. down plane. **10.** ·398 ft. per sec. per sec. **11.** 100 f. p. s. units. **12.** 39,420,000 f. p. s. units. **13.** 1·42 ft. per sec. **14.** 2464 f. p. s. units; 15 m. p. h. **15.** No. **18.** 204 lbs. wt. **19.** 19·13 secs. **20.** 280·5 yds. **22.** 1·61 ft. per sec. **23.** 33·6 : 1; ·42 : 1. **24.** ·935 ft. per sec. **25.** 2·78 m. p. h. **27.** 2190 ft. per sec. **28.** 5·6 m. p. h.; 323,800 ft.-pdals. **29.** 600 pdals; 18·6 lbs. wt. **31.** 3890 pdals; 162 ft.-pdals; the same. **32.** 27·3 f. p. s. units; 17 lbs. wt.

Chapter VII. Miscellaneous Exercises.

1. 189 tons wt. **2.** 1·6 ft. per sec. backwards; 4·04 tons wt. **3.** 388 lbs. wt. **4.** 5 ft. 8 ins. **5.** 1·75 lbs. wt. **6.** 20·9 lbs. wt. **7.** 9 ft. per sec. **8.** 7·28 lbs. wt. **9.** 800 ft. per sec. **11.** 1420 ft. per sec. **12.** 128 f. p. s. units; ·994 lb. wt. **13.** 494 cm. per sec. **14.** 2364 lbs. wt. **15.** 14·9 lbs. wt.; 13·3 lbs. wt. at 43° 54′ to pitch. **16.** 382·5 lbs. wt. **17.** 459·1 lbs. wt. **18.** ·0267 sec. **19.** 170·5 tons wt. **20.** Av. speed = 32·7 ft. per sec.; mean of initial and final speeds = 44 ft. per sec. **21.** 54,450, 73,160 pdals. **22.** $M \times \cdot 1633$, $M \times \cdot 1215$ pdals; no; yes. **23.** 2·125 f. p. s. units; ·198 ft.-lbs. **24.** 212·7 ft. **25.** 144 f. p. s. units, N. 46° E. **26.** ·35 m. p. h. **27.** ·7065, ·898, 1·006 ft. per sec. **28.** 2650 ft. per sec. **29.** 4·06 f. p. s. units; 203 pdals at 6° 6′ with normal. **30.** 216,800 pdals; 13,470 lbs. wt.

Chapter VIII. Examples.

1. 8·4 lbs. wt. **2.** 284·4 lbs. wt. **3.** 19·4 lbs. wt. **4.** 2·16 H. P. **5.** 1·28 lbs. wt. **6.** 780 sq. ft. **7.** ·273 lb. wt. **8.** 212·7 lbs. wt. **9.** 23,400 ft.-lbs. **10.** 42·5 H. P. **12.** ·00186 lb. wt. **13.** ·00979 lb. wt. **14.** ·00002 lb. wt.; 20°. **15 and 16.** 63·1 ft.-lbs. **17.** 2820 f. p. s. units. **18.** 14·1 pdals. **19.** ·796 H. P. **20.** 14,100 ft.-pdals; 1833 ft. per sec. **21.** 62·6 %; 1·25 times. **22.** 1·19 H. P.; 21,150 ft.-pdals; 735 ft. per sec.; 94 %; 1·88 times. **24.** 182·1 cub. ft.; 155,100 f. p. s. units; 4815 lbs. wt.; 11·6 sq. ft. (taking mass of 1 cub. ft. of sea water as 64 lbs.).

Chapter VIII. Miscellaneous Exercises.

1. 5000 f. p. s. units; 2·59 lbs. wt. **2.** 4503 lbs. wt. **3.** 1154 lbs. wt. **4.** 72·7 lbs. wt. **5.** 259 lbs. wt. **6.** 96·2 lbs. wt.; 7·7 H. P. **7.** ·235 ft. per sec. **8.** 3·66 lbs. wt. at 70° to jet. **9.** 795·6 lbs.; 67·3 m. p. h. **10.** 978, 525 lbs. wt. **11.** ·066 lbs. wt., bisecting angle. **12.** 211·4 H. P. **13.** 2·09 lbs. wt.; 67° 42′; 1·32 lbs. wt. **15.** 172 sq. ft. **16.** 441 lbs. wt.; 83·5 m. p. h. **17.** 469·8 lbs. wt.; 7·5 H. P.

Chapter IX. Examples.

1. 4·1, 6·9, 8·1, 5·5, 3·9, 1·1, $-$3·4, ft. per sec. **2.** 0, ·375 sec.; ·17 sec.; no. **3.** 640, 1220, 1760 ft. per sec.; 1800, 1000 ft. per sec.; no. **4.** 1·5, 1·15, ·708 ft. per sec. per sec. **5.** 72, $-$44 ft. per sec. per sec. **6.** 67 lbs. wt. **7.** ·186 lb. wt. **8.** 11·25 ft. **9.** 1538 ft. **11.** 35·1 ft. **13.** 41° 42′. **14.** 178 lbs. wt. **15.** 1 in 1·5. **16.** 1 in 1·92. **17.** 57 lbs. wt. **18.** 9·1 ins. **19.** Inner, 7605 lbs. wt. vertical; outer, 9195 lbs. wt. vertical, 1260 lbs. wt. horizontal. **20.** 8·025 ft. per sec. **21.** 17·94 ft. per sec. **22.** 1·66 secs.

Chapter IX. Miscellaneous Exercises.

1. 8, 11·36 ft. per sec. **2.** 3·75, 10·67 ft. per sec. **3.** ·43 sec. **4.** 13° 57′. **5.** 27·4 miles an hr. **6.** 4 ft. per sec. **7.** 12·24 m. p. h. **8.** ·003466 lb. wt. **9.** ·79 ft. per sec. per sec.; 3·07 tons wt. **10.** 63 ft. **11.** 6·6 secs.; 10·7 pdals. **12.** 113 ft. **13.** ·535 lb. wt.; 17·13 ft. per sec.; 3·54 lbs. wt. **14.** 3078 lbs. wt. **15.** 154 ft.-lbs.; 8·1 ft. per sec. **16.** 150 lbs. wt.; 225,000 f. p. s. units; 16·7 m. p. h. **17.** 7500, 23,200, 13,800, 23,700 ft.

(The equation is $S = 23700 + 72500 \log \cos \left(\frac{724 - t}{11\cdot 7}\right)^\circ$ ft.)

18. ·197, ·073 ft. per sec. per sec.; 13·7, 5·05 lbs. wt. per ton; $a = 4\cdot 9$, $b = \cdot 00475$ (but probable error is large). **19.** 8·9, 12·2, 14·8 ft. per sec.; 64, 170, 305 ft. **20.** ·74, ·25 ft. per sec. per sec.; 322, 109 tons wt.

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CAMBRIDGE: PRINTED BY JOHN CLAY, M.A. AT THE UNIVERSITY PRESS